

# Mathematical Models for the Dynamics of Tobacco Use, Recovery, and Relapse

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## Abstract

A general epidemiological model is presented to describe the dynamics of drug use among adolescents, specifically tobacco use. Specific models are derived by considering other factors that have been identified to have an effect on the growing trend of tobacco use. The factors considered are peer pressure, relapse, counseling and treatment.

## Introduction

In 1994, the Surgeon General's Report on the prevention of tobacco use among young people concluded that tobacco use usually starts before high school graduation; that most young smokers are addicted to nicotine with similar symptoms as those experienced by adults; that the use of other drugs such as marijuana and alcohol is often preceded by the use of tobacco; that peer pressure exerts a great amount of influence on those with lower levels of achievement and self-esteem; that tobacco advertising increases the risk of smoking, and that efforts such as tax increases, prevention programs and media campaigns are successful in reducing the use of tobacco among youth. These conclusions indicate the importance of health policies directed to the younger population in order to diminish the consequences of tobacco use and abuse. The report strongly suggests the epidemic nature of tobacco use<sup>1</sup>.

Among the different definitions of drug addiction adopted by health organizations, a core of criteria is clearly identifiable. Some of these criteria are the compulsive use of a drug despite damage to the individual or to society, temporarily gratifying physical dependence, progressive tolerance of the drug's effect with increasing dosage intake, and strong tendency to relapse after use abstinence. An extensive amount of research demonstrates the addictive character of nicotine. Furthermore, the mechanisms that lead to tobacco addiction are similar to those that lead to addiction of other drugs<sup>2</sup>.

According to a report of the Centers for Disease Control and Prevention, the prevalent use of tobacco remains the leading preventable cause of death in the United States, causing a great amount of loss in human resources. Tobacco use causes more than 400,000 deaths each year with related costs of more than 50 billion dollars<sup>3</sup>. Although a direct causation has not been established, nicotine effects in long-term smokers include cardiovascular disease, hypertension, reproductive disorders, cancer and gastrointestinal disorders. These and other consequences related to tobacco use justify its inclusion among the most serious public health problems.

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<sup>1</sup>Elders M, Perry C, Eriksen M, Giovino G (1994). Commentary: The Report of the Surgeon General: Preventing Tobacco Use Among Young People, *American Journal of Public Health* 84(4): 543-547.

<sup>2</sup>*The Health Consequences of Smoking: Nicotine Addiction*. (1988). A Report of the Surgeon General, Washington, DC: US Department of Health and Human Services, Centers for Disease Control. DHHS publication CDC 88-8406.

<sup>3</sup>CDC Report on Tobacco Use, January 25, 1996.

Exerting more pressure for immediate intervention and public policy-making are the reported increases in tobacco and drug use by young people. The 1996 Monitoring the Future Survey, that monitors trends in drug use among Americans, found that since 1991, current smoking has increased from 14.3 to 21.0 percent for eighth graders, from 20.8 to 30.4 percent for tenth graders and from 28.3 to 34.0 percent for twelfth graders. The 1995 National Household Survey on Drug Abuse estimated that from 61 million Americans that were current smokers, 4.5 million were adolescents between 12 and 17 years of age. The Surgeon General reports that in 1991 the average age when smokers tried a cigarette for the first time was 14.5 years, and the average age when they became daily smokers was 17.7 years. The observed trend from 1975 to 1992 of relevant data confirm that adolescence is the primary time during which tobacco use develops<sup>4</sup>. The epidemic character of this onset at the early stages of youth deems it useful to approach the problem using dynamic mathematical models.

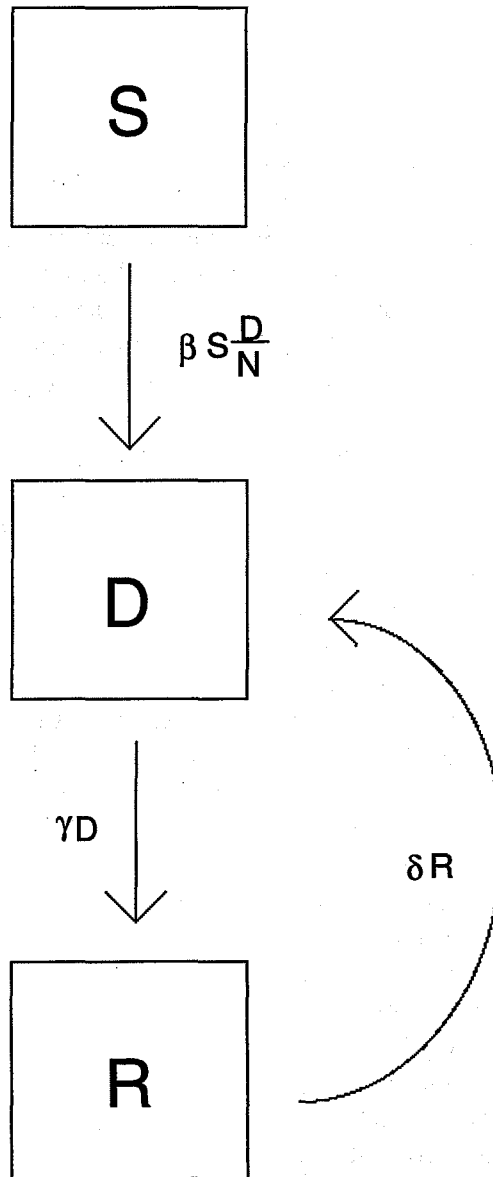
### General Mathematical Model for Drug Abuse

We propose a general model for drug abuse that can easily be specified to study the dynamics of tobacco use. We consider three classes of individuals in a constant population size of  $N$ . The class  $S$  is composed by those individuals that are susceptible to becoming regular drug users. The class  $D$  is composed by individuals who are regular drug users, meaning that they consume a considerable amount of drug in an habitual manner. Individuals that are treating themselves in any form or that have recovered from habitual drug use are included in the class  $R$ . We take  $c$  as the per capita visit rates to social gatherings per unit of time;  $cS$  gives the average number of visits to social gatherings of the susceptible population per unit of time. Susceptible individuals gather with  $S$ ,  $D$  and  $R$  individuals. Out of those gatherings what may influence a susceptible individuals is the presence of drug users given by the proportion  $D/N$ . Therefore, the total average visit (influence) rate is  $cSD/N$ . But only a proportion of those visits is effective in triggering drug use. Taking  $p$  as the probability that an influence (presence of drug users) results in the use of a drug, and  $\varphi$  as the proportion of individuals that become habitual users after a casual drug use, then the incidence or

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<sup>4</sup>Johnston LD, O'Malley PM, Bachman JG (1992). *Smoking, Drinking, and Illicit Drug Use among American Secondary School Students, College Students, and Young Adults, 1975-1992*. Volume 2: *College Students and Young Adults*. Bethesda, Md: US Department of Health and Human Services, National Institutes of Health; NIH publication 93-3481.

# General Mathematical Model For Drug Abuse



effective influence rate (new habitual drug users per unit of time) is  $\beta SD/N$ , where  $\beta = \varphi pc$  becomes the per capital effective influence rate. We take  $\gamma$  as the recovery rate per habitual drug user per unit of time, and  $\delta$  is the relapse rate per recovered individual per unit of time. The incidence or effective influence rate can be seen as the effective visit rate to drug users' niches, where drug users' niches are any kind of social gathering where drug use is present. The model is the following:

$$\begin{aligned}\frac{dS}{dt} &= -\beta S \frac{D}{N} \\ \frac{dD}{dt} &= \beta S \frac{D}{N} - \gamma D + \delta R \\ \frac{dR}{dt} &= \gamma D - \delta R\end{aligned}$$

This is a simple relapse model without vital dynamics where susceptible individuals become regular drug users mainly through peer pressure.

### Peer Influence

Peer pressure is without doubt heavily present in American adolescents, but there are important questions about the details, the magnitude and the mechanisms that define peer influence<sup>5</sup>. According to B. Bradford Brown, there is a misguided perception that peer pressure is a direct and transparent phenomenon. Brown points to empirical evidence that suggests that peer pressure acts on an individual in a multidimensional manner where normative and interactional influences, and the adolescents' willingness to be influenced are the most important factors. He mentions four key findings in the effort to measure peer influence. First, susceptibility to peer influence among young people is not uniform. In fact, susceptibility reaches its higher level in the younger population and in people with low confidence and lack of social interaction skills. Second, adolescents cannot clearly identify situations where peer pressure is explicit. Third, peer pressure does not dictate adolescents' behavior. Finally, susceptibility to peer influence varies according to family structures and parental guidance.

Our model presupposes the effects of peer pressure in the dynamics of drug use. As presented, it only considers peer influence in the recruitment of new drug users (the nonlinear term). The relapse component of the

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<sup>5</sup>Brown BB. (1993). Peer Groups and Peer Culture. In: *At the Threshold: The Developing Adolescent*. Harvard University Press, ed. Feldman S, Elliot G.

model, however, does not explicitly describe the mechanisms of the relapse phenomenon. It only states that a proportion of the temporarily recovered individuals relapse at a constant rate.

### Analysis of the General Model with Vital Dynamics

We extend the model just presented to include demographic factors in the population considered. As previously stated, we assume that the population remains constant, i.e., every death is balanced with a birth in the susceptible population. We take  $\mu$  as the constant mortality rate. The model is as follows:

1.  $\frac{dS}{dt} = \mu N - \beta S \frac{D}{N} - \mu S.$
2.  $\frac{dD}{dt} = \beta S \frac{D}{N} - \gamma D + \delta R - \mu D.$
3.  $\frac{dR}{dt} = \gamma D - \delta R - \mu R.$

This system can be reduced to the following:

- (1')  $D' = \beta(N - D - R) \frac{D}{N} - \gamma D + \delta R - \mu D.$
- (2')  $R' = \gamma D - \delta R - \mu R.$

### Disease-free state and basic reproductive number

The disease-free equilibrium is  $\bar{X} = (N, 0, 0)$ . To analyze the stability of  $\bar{X}$  we compute the basic reproductive number. Consider (2) and (3),  $S \approx N$ ,

$$0 = \beta S \frac{D}{N} - \gamma D + \delta R - \mu D \quad \text{and} \quad 0 = \gamma D - \delta R - \mu R.$$

Considering the linear terms of the corresponding variable in each equation,

$$D = \frac{\beta}{\gamma + \mu} D + \frac{\delta}{\gamma + \mu} R \quad \text{and} \quad R = \frac{\gamma}{\delta + \mu} D.$$

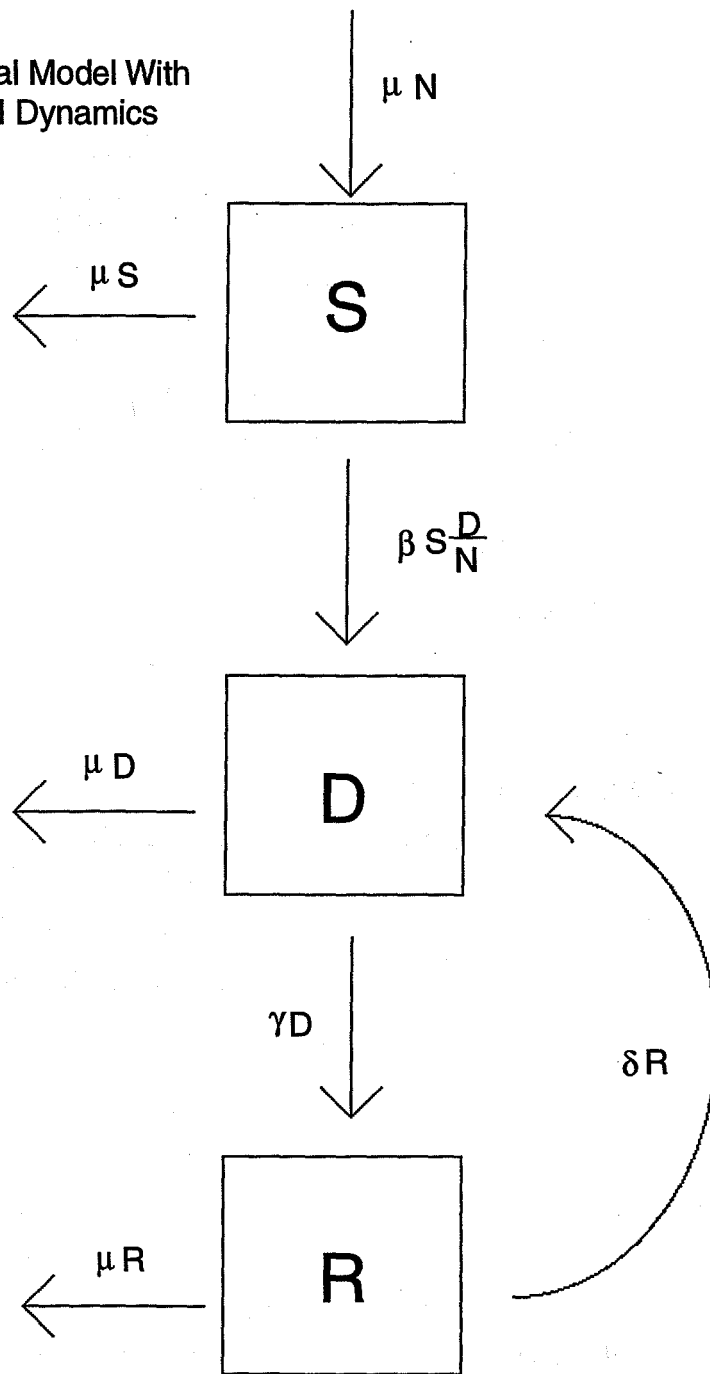
The Jacobian at (0,0) is the following:

$$J_0 = \begin{pmatrix} \frac{\beta}{\gamma + \mu} & \frac{\delta}{\gamma + \mu} \\ \frac{\gamma}{\delta + \mu} & 0 \end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - \frac{\beta}{\gamma + \mu} \lambda - \frac{\gamma \delta}{(\delta + \mu)(\gamma + \mu)} = 0,$$

General Model With  
Vital Dynamics



from which we obtain

$$\lambda = \frac{1}{2} \left( \frac{\beta}{\gamma + \mu} \right) \pm \frac{1}{2} \sqrt{\frac{\beta^2}{(\gamma + \mu)^2} + \frac{4\gamma\delta}{(\delta + \mu)(\gamma + \mu)}}.$$

We take the dominant eigenvalue as the basic reproductive number,

$$R_0 = \frac{1}{2} \left( \frac{\beta}{\gamma + \mu} \right) + \frac{1}{2} \sqrt{\frac{\beta^2}{(\gamma + \mu)^2} + \frac{4\gamma\delta}{(\delta + \mu)(\gamma + \mu)}}.$$

It follows that the disease-free equilibrium is stable iff  $R_0 < 1$ . Note also that

$$R_0 = \left( \frac{\beta}{\gamma + \mu} \right) \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\gamma\delta(\gamma + \mu)}{\beta^2(\delta + \mu)}} \right]$$

and if  $R'_0 = \frac{\beta}{\gamma + \mu}$ , then

$$R'_0 > 1 \text{ implies } R_0 > 1.$$

### Endemic Equilibria and Stability Analysis

It can be seen from equations (1') and (2') that if either  $R = 0$  or  $D = 0$  we obtain (0,0). For a state of coexistence, where  $D > 0$ ,  $R > 0$ , we analyze (1') and (2'),

$$0 = \beta D - \beta \frac{D^2}{N} - \beta R \frac{D}{N} - \gamma D + \delta R - \mu D \quad \text{and} \quad 0 = \gamma D - \delta R - \mu R,$$

from which we obtain the following solutions:

$$D = \frac{\mu + \delta}{\mu + \delta + \gamma} - \frac{\mu}{\beta} \quad \text{and} \quad R = \frac{\gamma}{\mu + \delta + \gamma} - \frac{\mu\gamma}{\beta(\mu + \delta)}.$$

Note that for  $D$  and  $R$  to be in the first quadrant, the following condition must hold:

$$\frac{\beta(\mu + \delta)}{\mu(\mu + \delta + \gamma)} > 1.$$

For the stability analysis of  $X^* = (D^*, R^*)$  we compute the Jacobian at this point. The general Jacobian is

$$J = \begin{pmatrix} \beta - 2\beta \frac{D}{N} - \beta \frac{R}{N} - \gamma - \mu & -\beta \frac{D}{N} + \delta \\ \gamma & -\delta - \mu \end{pmatrix}.$$



Evaluating  $J$  at  $X^*$  is

$$J_{X^*} = \begin{pmatrix} \frac{1}{N} \left[ \frac{\beta(\gamma-2\beta)}{\delta+\gamma+\mu} + \frac{\gamma\mu}{\delta+\mu} + 2\mu \right] + \beta - \mu - \gamma & \frac{1}{N} \left[ \frac{-\beta(\mu+\gamma)}{\mu+\delta+\gamma} + \mu \right] + \delta \\ \gamma & -\delta - \mu \end{pmatrix}.$$

For  $X^*$  to be stable the following must hold:

$$\text{trace } J_{X^*} < 0 \Leftrightarrow \frac{2\beta - \mu}{(\mu + \delta)(\mu + \delta + \gamma)} + \frac{\delta}{\beta} \frac{(\delta + \mu + \gamma)}{(\delta + \mu)} > 1,$$

$$\det J_{X^*} > 0 \Leftrightarrow \frac{2\beta}{(\mu + \delta + \gamma)} - \frac{\mu}{\beta} \frac{(\delta + \mu + \gamma)}{(\delta + \mu)} > 1.$$

Graphical methods seem more appropriate for the stability analysis of this endemic state.

### Parameter Estimation

In the simplest case (not quite realistic)

$$\frac{\partial D}{\partial t} = -\gamma D.$$

Then

$$\frac{D(t)}{D(0)} = e^{-\gamma t}.$$

At year  $t = 1$

$$\frac{D(1)}{D(0)} = e^{-\gamma}.$$

$1 - [D(1)/D(0)]$  is the proportion of people that leave  $D$  in one year. We consider the following information in the report of the Surgeon General:

- 73% of smokers in one year try to quite smoking.
- 81% of these are unsuccessful.

Therefore

- 59% of smokers relapse in one year.
- 14% of smokers recover in one year.

Therefore,

$$1 - \frac{D(1)}{D(0)} = 0.14, \quad 1 - e^{-\gamma} = 0.14, \quad e^{-\gamma} = 0.86$$

and

$$\gamma = -\log 0.86 \approx 0.15.$$

Then the average time as a smoker ( $D$ ) is

$$\frac{1}{\gamma} = \frac{1}{0.15} = 6.\bar{6} \text{ years.}$$

With similar assumptions,

$$\frac{\partial R}{\partial t} = -\delta R \quad \text{and} \quad \frac{R(t)}{R(0)} = e^{-\delta t}.$$

$1 - [R(1)/R(0)]$  is the proportion of people that leave  $R$  in one year. Using the previous information,

$$1 - e^{-\delta} = 0.59 \quad \text{and} \quad \delta = -\log 0.41 \approx 0.89.$$

Then the average time before relapse is

$$\frac{1}{\delta} = \frac{1}{0.89} \approx 1.12 \text{ years.}$$

When  $R_0 > 1$  the system has a unique equilibrium,

$$S \rightarrow S_\infty, \quad D \rightarrow D_\infty, \quad \beta S \frac{D}{N} \rightarrow \beta S_\infty \frac{D_\infty}{N}.$$

If we take  $\alpha = \beta (D_\infty/N)$  then near equilibrium  $(dS/dt) = \alpha S$ . Using the information provided, the average time before becoming a regular smoker is 2.5 years. Therefore

$$\frac{1}{\alpha} = 2.5 \quad \text{and} \quad \alpha = 0.4.$$

This is the case since we are considering the population of young people from 12 years of age to 22 years of age. From this fact we can set the mortality rate  $\mu = 0.10$ . If we define

$$A = \frac{1}{\alpha} = \frac{N}{\beta D_\infty}, \quad L = \frac{1}{\mu}, \quad \tau = \frac{1}{\gamma},$$

then

$$\begin{aligned}R_0 &\approx \frac{\beta}{\gamma + \mu} = \beta \tau \frac{L}{\tau + L} = \frac{N}{S_\infty}, \\S_\infty &\approx \frac{\tau + L}{\beta \tau L} N = \frac{N}{R_0}, \\D_\infty &\approx \frac{N}{BL S_\infty} (N - S_\infty).\end{aligned}$$

Also,

$$\frac{L}{A} = L \frac{\beta D_\infty}{N} = \frac{N}{S_\infty} - 1 = R_0 - 1.$$

Therefore

$$R_0 \approx 1 + \frac{L}{A} = 1 + \frac{\gamma}{\mu} = 1 + \frac{0.15}{0.10} = 2.5.$$

From this approximate value of  $R_0$  we obtain  $\beta \approx 0.625$ . Our list of parameters is the following:

$$\beta = 0.625, \quad \delta = 0.89, \quad \gamma = 0.15, \quad \mu = 0.10, \quad R_0 = 2.5.$$

## Simulations of General Model with Vital Dynamics

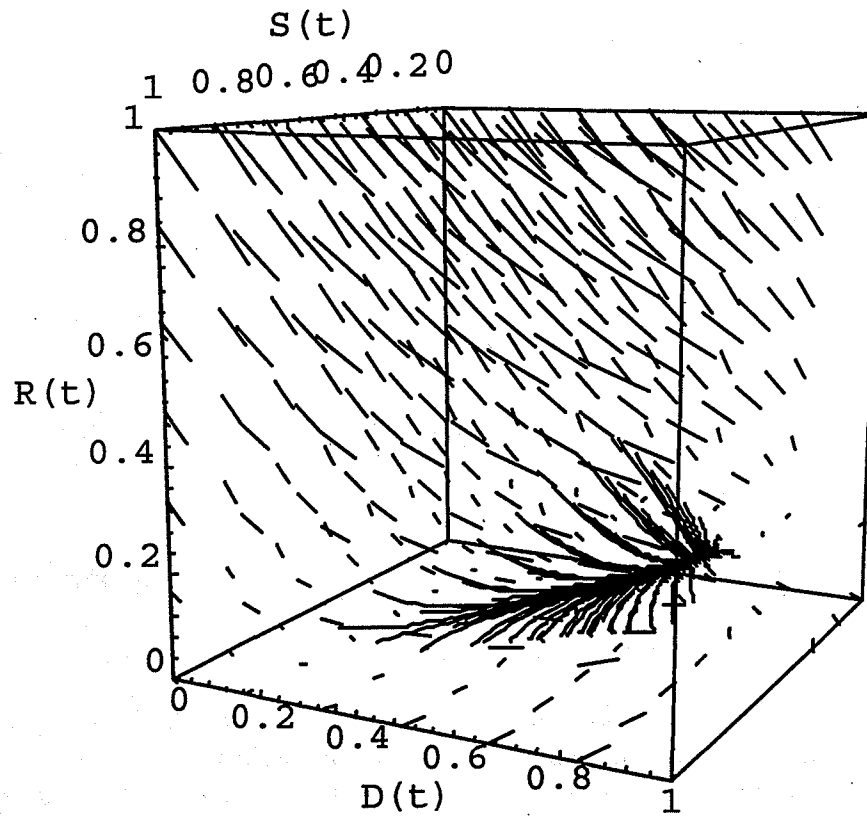
### Three-dimensional Parametric Plot of All Classes

The simulations were run with the parameters estimated in the previous section. The first graph shows the general pattern of flows and the vector field. As can be seen, all trajectories under any valid initial conditions tend toward an endemic point which is clearly stable.

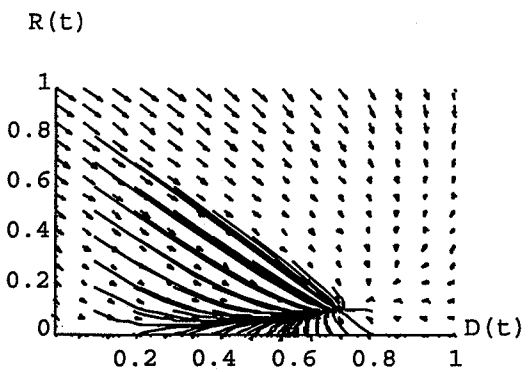
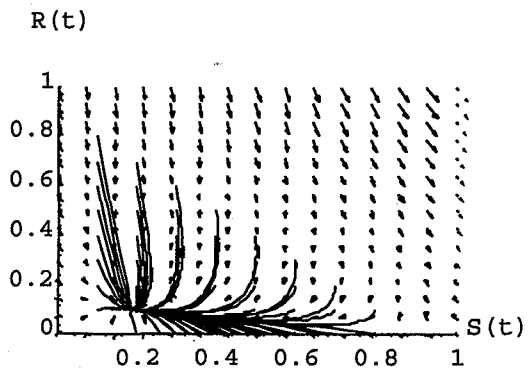
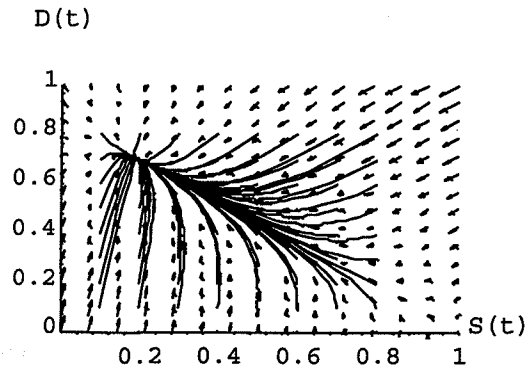
Graphs on following pages

# Simulations of General Model with vital dynamics

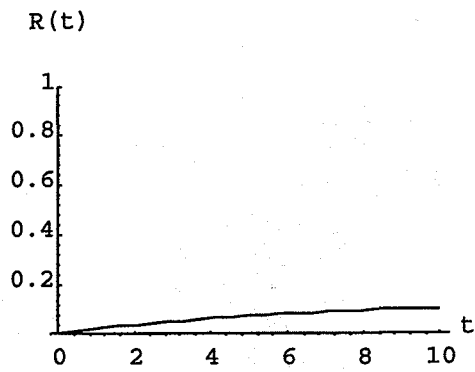
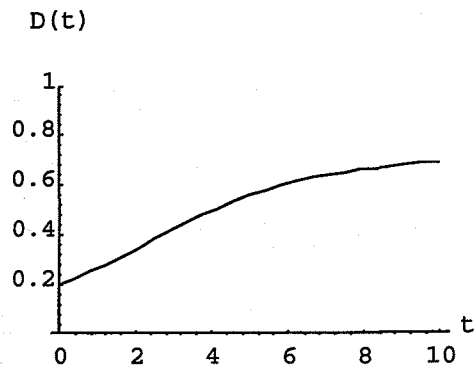
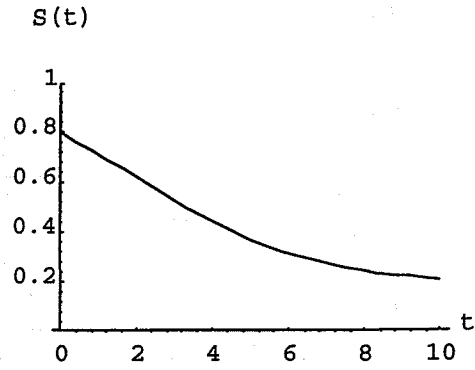
## Three dimensional parametric plot of all the classes



**Two dimensional parametric plots for every pair of classes**



### Population Growth of each Class with respect to time



## Remarks on the General Model

As can be noted from the mathematical analysis, with the estimated parameters, smoking is an endemic social problem with a rate of infectivity of two to three persons per smoker, but this is assuming that there is no systematic rehabilitation program nor any operational parameter to drive away the system from this trend. The great effort public health agencies are putting to resolve the problem of tobacco among adolescents is a factor that will definitely disturb the observed growing trend.

It is evident that for tobacco use to cease to be an endemic problem, the effective influence rate per individual and/or the recovery rate must be significantly increased. Since the (internal) mechanisms of peer pressure are just beginning to be elucidated more clearly, factors that determine the recovery rate are the most operational at the moment, thus the need for external measures.

## Nonlinear Relapse Model

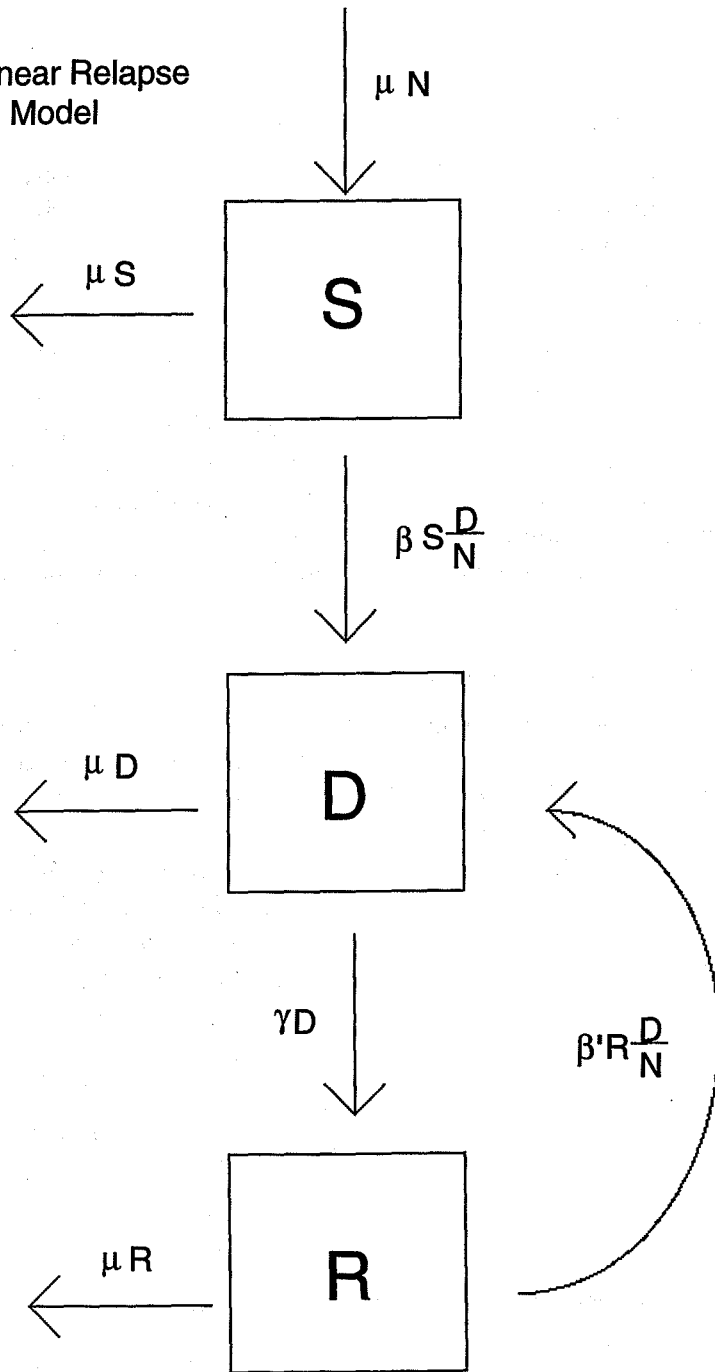
The previous model considered the recruitment of susceptible individuals into the class of regular drug users as an effect of peer pressure. Nevertheless, the model did not specify clearly the mechanism for the relapse process. The present model considers a similar peer pressure mechanism for the relapse of temporarily recovered individuals. This is done by taking  $\delta = \beta' \frac{D}{N}$  as a parameter dependent on the presence of drug users, and taking  $\beta'$  to be the per capita effective influence rate for the relapse to drug use.  $\beta'$  is defined similarly as  $\beta' = \varphi' p' c'$ . The model is specified by the following equations:

$$\begin{aligned}\frac{dS}{dt} &= \mu N - \beta S \frac{D}{N} - \mu S, \\ \frac{dD}{dt} &= \beta S \frac{D}{N} - \gamma D + \beta' R \frac{D}{N} - \mu D, \\ \frac{dR}{dt} &= \gamma D - \beta' R \frac{D}{N} - \mu R, \\ N &= S + D + R.\end{aligned}$$

This model can be reduced to the following equations:

1.  $\frac{dD}{dt} = \beta D - \beta \frac{D^2}{N} - \beta R \frac{D}{N} + \beta' R \frac{D}{N} - \mu D - \gamma D.$
2.  $\frac{dR}{dt} = \gamma D - \beta' R \frac{D}{N} - \mu R.$

Non-linear Relapse Model





Basic Reproductive Number and stability of  $\bar{X} = (0, 0)$ .

The general Jacobian is the following:

$$J = \begin{pmatrix} \beta - 2\beta \frac{D}{N} - \beta \frac{R}{N} + \beta' \frac{R}{N} - \gamma - \mu & -\beta \frac{D}{N} + \beta' \frac{D}{N} \\ \gamma - \beta' \frac{R}{N} & -\beta' \frac{D}{N} - \mu \end{pmatrix}.$$

Evaluated  $\bar{X}$

$$J(\bar{X}) = \begin{pmatrix} \beta - \gamma - \mu & 0 \\ \gamma & -\mu \end{pmatrix}.$$

For all eigenvalues to be negative the following must hold:

$$\beta - \gamma - \mu < 0 \Rightarrow \frac{\beta}{\gamma + \mu} < 1,$$

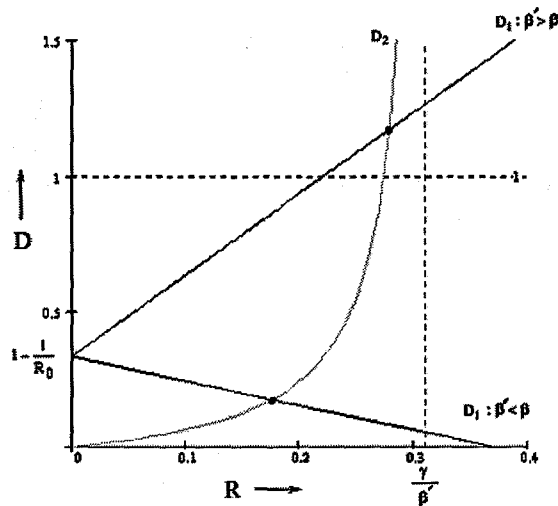
therefore the Basic Reproductive Number is  $R_0 = \frac{\beta}{\gamma + \mu}$ .

**Endemic equilibria and stability analysis**

If either  $D = 0$  or  $R = 0$ , we obtain  $\bar{X}$ . For  $D > 0$  or  $R > 0$  from (1) and (2) we obtain

$$D_1 = \frac{(\beta' - \beta)}{\beta} R + \frac{(\beta - \gamma - \mu)}{\beta} N \quad \text{and} \quad D_2 = \frac{\mu \frac{R}{N}}{\gamma - \beta' \frac{R}{N}}.$$

If  $R_0 > 1$  and  $\frac{\beta - \gamma - \mu}{\beta} > \frac{\mu}{\gamma}$  there exists an endemic equilibrium



If  $\frac{\beta-\gamma-\mu}{\beta} < \frac{\mu}{\gamma}$  and  $\beta' < \beta$ , there is no endemic equilibrium. If  $\frac{\beta-\gamma-\mu}{\beta} < \frac{\mu}{\gamma}$  and  $\beta' > \beta$ , simulations suggest that there is not an endemic equilibrium. For the stability of this point, see graphs.

### Estimation of $\beta'$

Assume the following:

$$\frac{dR}{dt} = -\beta' R \frac{D}{N}.$$

Take  $\psi = \beta' \frac{D}{N}$ . Near equilibrium with  $R_0 > 1$

$$\frac{dR}{dt} = -\psi R.$$

Then  $\psi = \beta' \frac{D_\infty}{N} = 0.89$  as for  $\delta$ . Since  $A = \frac{1}{\alpha} = \frac{N}{\beta D_\infty}$ , then

$$\beta' = 0.89 \frac{\beta}{\alpha} = (0.89) \frac{0.625}{0.4} \approx 1.4.$$

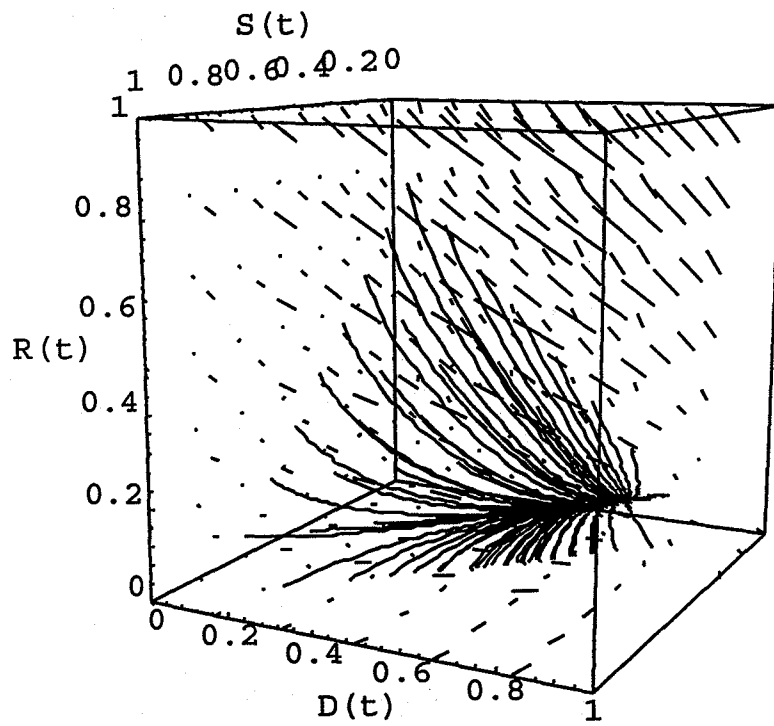
## Simulations of the Nonlinear Relapse Model

### Three-dimensional parametric plot of all classes

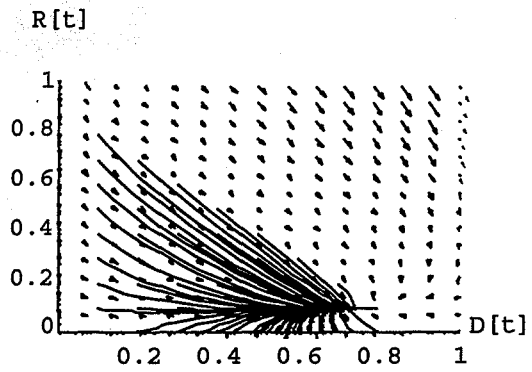
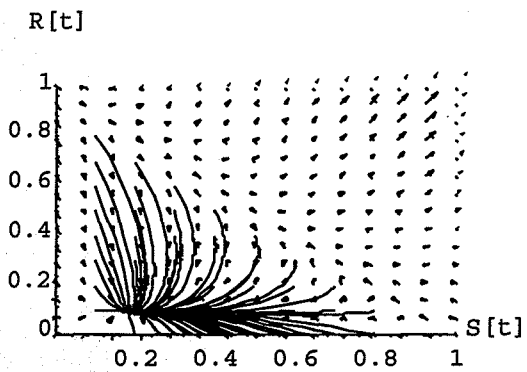
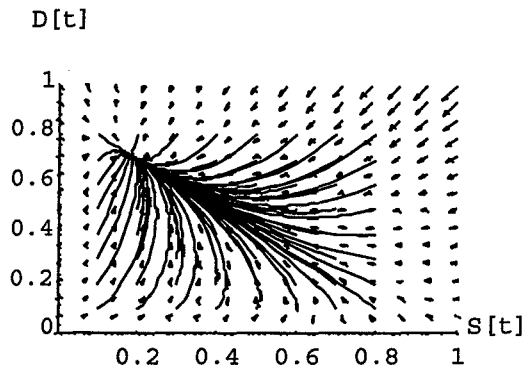
The simulations were run with the same parameters estimated in a previous section. This first graph shows the general pattern of flows and the vector field. As can be seen, all trajectories under any valid initial conditions tend toward an endemic point which is clearly stable. The nonlinear term that controls the change from class D to class R apparently increases slightly the proportion of individuals in the class D as time increases.

## Simulations of the Nonlinear Relapse Model

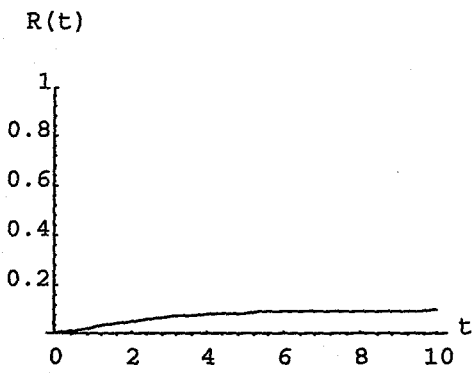
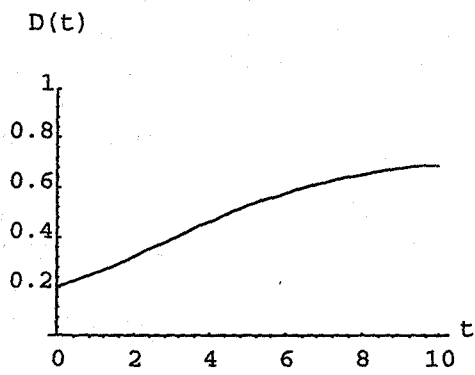
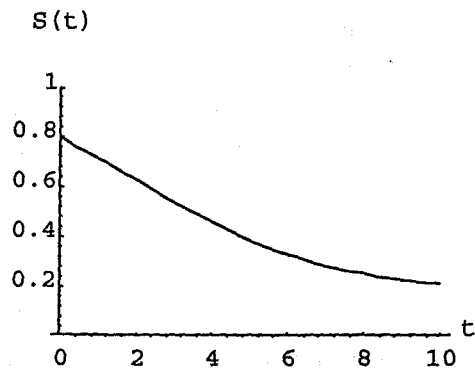
Three dimensional parametric plot of all the classes



## Two dimensional parametric plots for every pair of classes



## Population Growth of each Class with respect to time



### Remarks on the Nonlinear Relapse Model

The assumption of nonlinearity in the relapse mechanism, with the estimated parameters, brings the system to an equilibrium with a higher proportion of smokers, as was expected. Nevertheless, the dynamics were clearly very similar, since the trajectories were of the same nature.

Again, in this model no systematic recovery effort was being introduced, as it will be in the next models.

### Relapse with Rehab Program Effectiveness

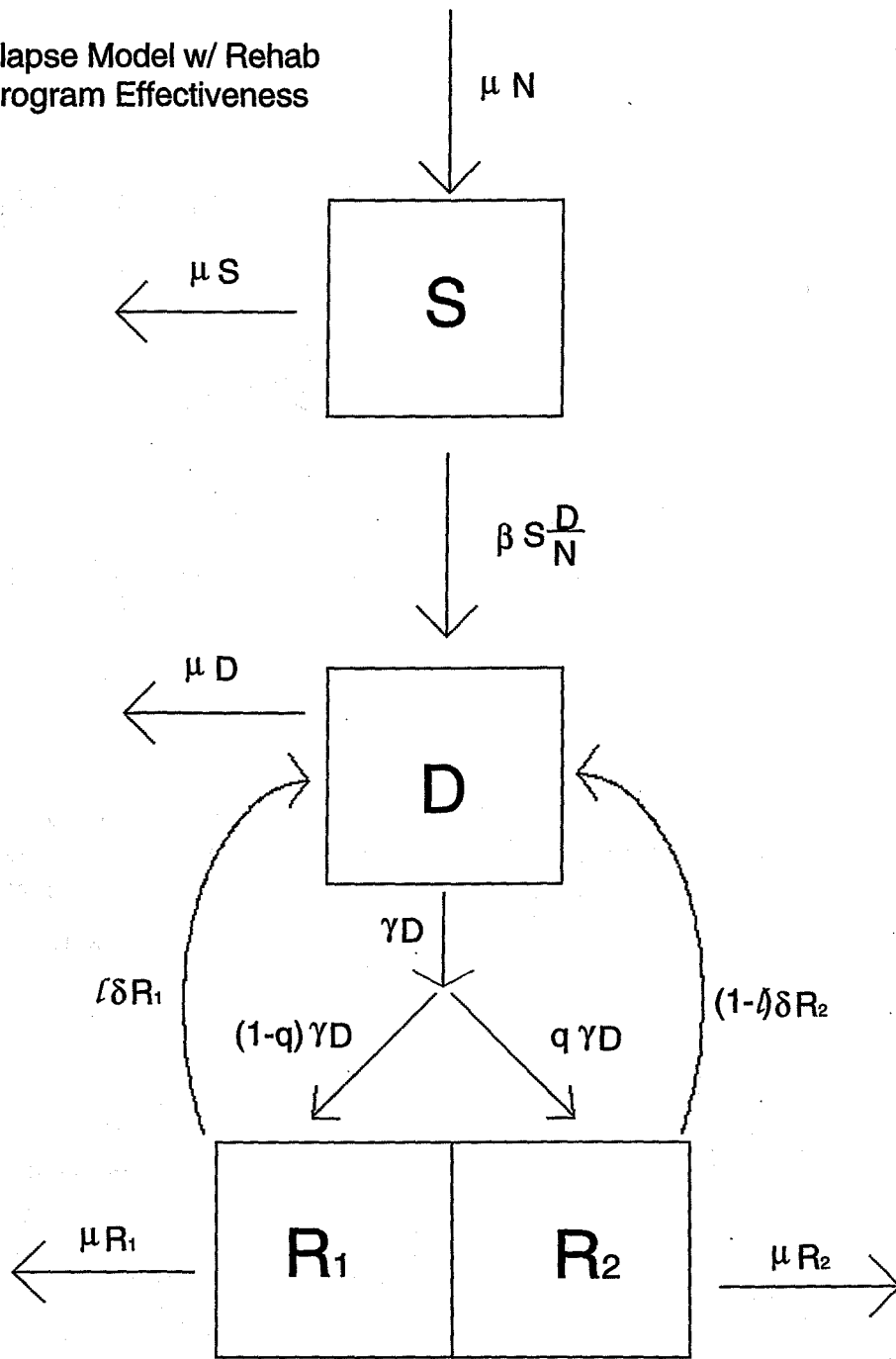
As we went on improving our model the question of the effectiveness of rehab programs arose, as well as the psychological awareness of the tobacco epidemic. We also wanted to know the proportion of recovered tobacco users that may relapse and the proportion of those that obtain complete rehabilitation. In order to assess this we consider the  $R$  class now as being divided into two different classes,  $R_1$  and  $R_2$ , where  $R_1$  are the recovered tobacco users that may relapse and  $R_2$  are the completely recovered tobacco users. A new parameter  $g$  is introduced to assess the effectiveness of rehab programs on the psychological awareness. From this assumption,  $g$  not only deals with the ineffectiveness of rehab programs but also with the proposed tobacco tax increase and strict legal measures that may effect the psychological and behavioral patterns of the recovered individuals.

$$\begin{aligned}\frac{dS}{dt} &= \mu N - \beta S \frac{D}{N} - \mu S, \\ \frac{dD}{dt} &= \beta S \frac{D}{N} - \gamma D + \delta R_1 - \mu D, \\ \frac{dR_1}{dt} &= (1-g)\gamma D - \mu \delta R_1 - \mu R_1, \\ \frac{dR_2}{dt} &= g\gamma D - \mu R_2 - (1-\mu)\delta R_2\end{aligned}$$

If we consider  $\ell \neq 1$  then the model reduces to the General Model with Vital Dynamics. Therefore, considering  $\ell \approx 1$ , and the fact that  $R = R_1 + R_2$ , and  $S = N - (D + R)$ , the system of four equations is reduced to a simpler two-equation system. See also that we can either have  $g$  as a constant or as a function of  $D$ , which would reflect the effect of peer pressure in rehab programs. First we consider the case of  $g$  being a constant.

1.  $\frac{dD}{dt} = \beta D - \frac{\beta D^2}{N} - \frac{\beta \gamma D^2}{N} - \gamma g D - \mu D.$
2.  $\frac{dR}{dt} = \gamma D - \gamma(1-g)D\delta - \mu R.$

Relapse Model w/ Rehab Program Effectiveness



Setting  $\frac{dD}{dt} = 0$ ,  $\frac{dR}{dt} = 0$ ,

$$D = D \left( \frac{\beta}{\gamma g + \mu} \right) - D^2 \left( \frac{\beta + \beta \gamma}{N(\gamma g + \mu)} \right) \quad \text{and} \quad R = D \left( \frac{\gamma - \gamma \delta + g \gamma \delta}{\mu} \right).$$

Evaluating the Jacobian at  $(0,0)$ ,

$$J(0,0) = \begin{pmatrix} \frac{\beta}{\gamma g + \mu} & 0 \\ \frac{\gamma - \gamma \delta + g \gamma \delta}{\mu} & 0 \end{pmatrix}.$$

$$\lambda^2 - \frac{\beta}{\gamma g + \mu} \lambda = 0,$$

$$\lambda \left( \lambda - \frac{\beta}{\gamma g + \mu} \right) = 0 \Rightarrow \lambda = 0 \quad \text{or} \quad \frac{\beta}{\gamma g + \mu} > 0,$$

$$R_0 = \frac{\beta}{\gamma g + \mu}.$$

### Equilibrium points

Again setting  $\frac{dD}{dt} = \frac{dR}{dt} = 0$  we obtain that if  $D = 0$  then  $R = 0$  and that if  $R = 0$  then  $D = 0$ . Therefore  $(0,0)$  is an equilibrium point. If  $R > 0$  and  $D > 0$  the equilibrium point of coexistence or endemic point is

$$\left( \frac{(\beta - \gamma g - \mu)N}{\beta + \beta \gamma}, \frac{(\beta - \gamma g - \mu)(\gamma - \gamma \delta + \gamma \delta g)N}{\beta + \beta \gamma} \right).$$

### Stability of Equilibria

At  $(0,0)$

The Jacobian Matrix of the reduced model evaluated at  $(0,0)$  is:

$$J(0,0) = \begin{pmatrix} \beta - \gamma g + \mu & 0 \\ \gamma - \gamma(1-g)\delta & -\mu \end{pmatrix}.$$

$$\lambda_1 = \beta - \gamma g - \mu, \quad \lambda_2 = -\mu,$$

$$R_0 = \frac{\beta}{\gamma g + \mu} < 1 \Leftrightarrow (0,0) \text{ stable.}$$



At  $(D, R)$

Evaluating the Jacobian Matrix of the reduced model at the endemic equilibrium we obtain:

$$J(D, R) = \begin{pmatrix} -(\beta - \gamma g - \mu) & 0 \\ \gamma - \gamma(1-g)\delta & -\mu \end{pmatrix}.$$

$$\det J(D, R) > 0 \quad \text{and} \quad \text{trace } J(D, R) < 0 \Rightarrow \\ \beta - \gamma g - \mu > 0 \Rightarrow R_0 > 1 \text{ for stability of } (D, R).$$

If  $g$  is a function of  $D$  of the form  $g(D) = \theta \left(1 - \frac{D}{N}\right)$  and taking on account the same assumptions of  $g$  being constant analysis, we reduce the model to two equations:

$$1. \quad \frac{dD}{dt} = \beta D - \gamma \theta D - \mu D + D^2 \left( \frac{\gamma \theta - \beta \gamma - \beta}{N} \right). \\ 2. \quad \frac{dR}{dt} = D(\gamma - \gamma \delta + \gamma \delta \theta) - D^2 \left( \frac{\gamma \theta \delta}{N} \right) - \mu R.$$

The Jacobian Matrix of this system at  $(0,0)$ :

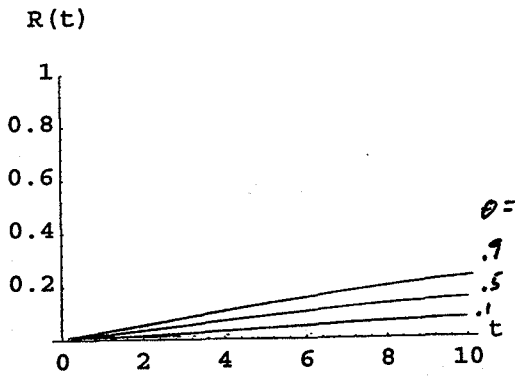
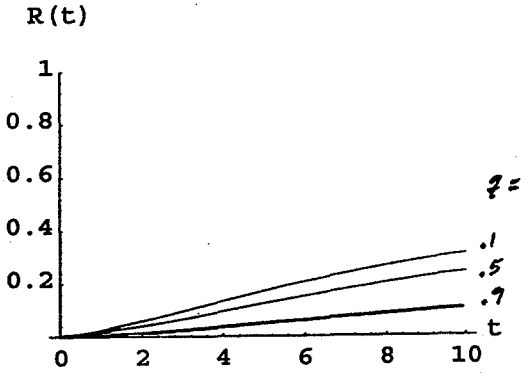
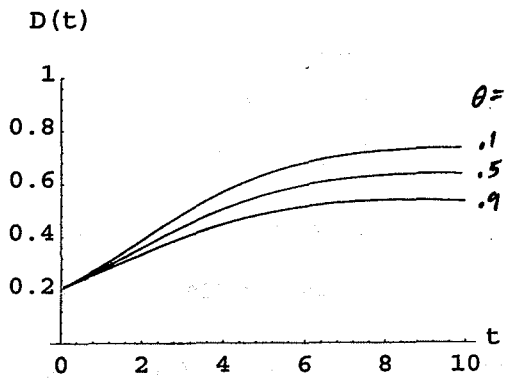
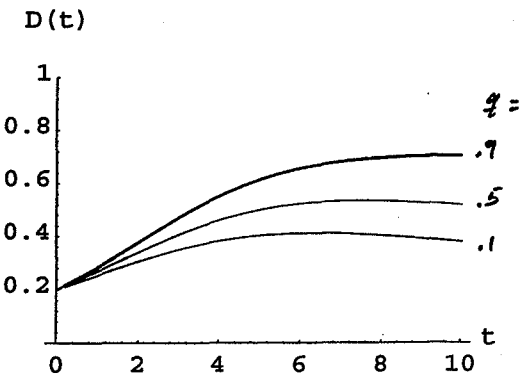
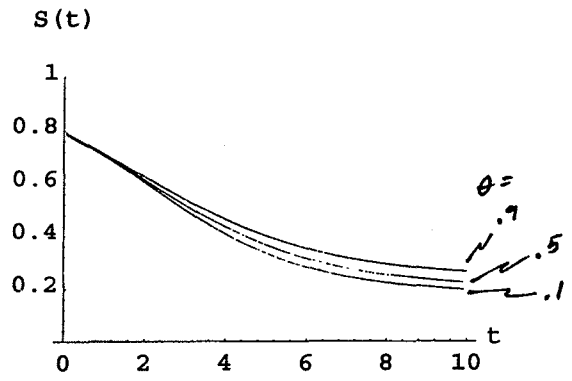
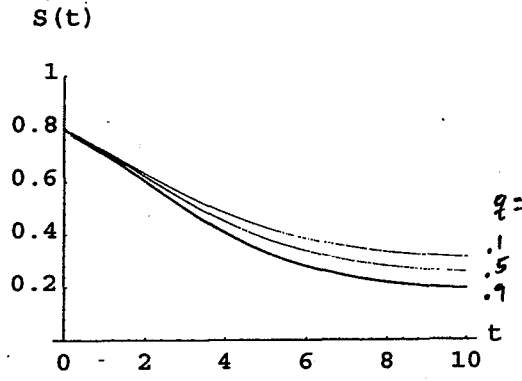
$$J(0,0) = \begin{pmatrix} \beta - \mu - \gamma \theta & 0 \\ \gamma - \gamma \delta + \gamma \delta \theta & -\mu \end{pmatrix}, \\ \lambda_1 = \beta - \mu - \gamma \theta, \quad \lambda_2 = -\mu, \\ R_0 = \frac{\beta}{\gamma \theta + \mu}, \\ (0,0) \text{ stable} \Leftrightarrow R_0 < 1.$$

If  $\frac{dD}{dt} = \frac{dR}{dt} = 0$ , the following condition for existence of the endemic equilibria is obtained:

$$\beta > \frac{\gamma \theta}{1 + \gamma} \quad \text{and} \quad R_0 > 1.$$

The endemic equilibria is reached at the point:

$$(D, R) = \left( \frac{(\mu + \gamma \theta - \beta)N}{\gamma \theta - \beta \gamma - \beta}, \right. \\ \left. N \left[ \frac{(\mu + \gamma \theta - \beta)(\gamma - \gamma \delta + \gamma \delta \theta)(\gamma \theta - \beta \gamma - \beta)^2 \gamma \theta \delta}{\mu(\gamma \theta - \beta \gamma - \beta)^2} \right] \right).$$



Evaluating the Jacobian Matrix at the endemic equilibrium:

$$J(D, R) = \begin{pmatrix} (\beta - \mu - \gamma\theta) - \left(\frac{\mu + \gamma\theta - \beta}{\gamma\theta - \beta\gamma - \beta}\right) (2\beta + 2\beta\gamma - 2\gamma\theta) & 0 \\ (\gamma - \gamma\delta + \gamma\delta\theta) - \left(\frac{\mu + \gamma\theta - \beta}{\gamma\theta - \beta\gamma - \beta}\right) (2\gamma\delta\theta) & -\mu \end{pmatrix}.$$

$$\det(D, R) > 0 \text{ and } \text{trace}(D, R) < 0 \Rightarrow R_0 = \frac{\beta}{\mu + \gamma\theta} > 1$$

for the stability of the point.

### Discussion and Conclusions

We considered a basic and general model for drug use in a population of adolescents, with peer pressure as a key ingredient in the recruitment mechanism of new drug users. We estimated the parameters of the model given some simplifying assumptions and determined a rough approximation of the basic reproductive number  $R_0$ . Based on these parameters, we performed some simulations that clearly showed the endemic character of tobacco use among adolescents.

A second model was presented, where not only the recruitment of new smokers was a consequence of peer pressure, but the relapse process as well. This model showed an increase in the population of smokers after the 10-year interval, as compared to the previous model.

Recent efforts by community, academic and health organizations to disturb the growing trend of tobacco use will significantly effect the dynamics of this health and social issue. A third model was presented to mimic the possible effect of counseling and treatment. Counseling was considered as an internal "immunization" and treatment as an external policy. The advantages of counseling was definite against external policies, but in a real context are the most difficult to implement. An interesting result was the fact that for the model where recovery efforts were effected by peer cultures (where  $g$  was a function of  $D$ ), the basic reproductive number was essentially the same, both dependent on the intrinsic effectiveness of the recovery effort. This suggests an important result: counseling efforts, or "cultural immunization" creates a class of individuals less likely to relapse and by no means susceptibles. These last set of models (with recovery efforts) also brought significant changes in the overall dynamics when seen against the previous ?.

Although this model may be a brute simplification of a complex problem, it clearly points to the importance of educational (preventive) measures *vis à vis* external measures against drug use and abuse.

### Parameters (definition and estimated value)

- $C$  = per capita visit rate to social gatherings  
per unit of time,
- $CS$  = average number of visits of susceptible individuals  
to social gatherings,
- $CD \frac{D}{N}$  = total average influence rate per unit of time,
- $P$  = probability that an influence (presence of drug users)  
results in the intake of drugs,
- $\varphi$  = probability that a casual user becomes  
an habitual user,
- $\beta = \varphi PC$  = per capita effective influence rate, (0.625)
- $\beta S \frac{D}{N}$  = incidence rate (new habitual drug users  
per unit of time),
- $\gamma$  = recovery rate per habitual drug user  
per unit of time, (0.15)
- $\delta$  = relapse rate per recovered individual  
per unit of time, (0.89)
- $\mu$  = per capita death rate, (0.10)