

The Effects of Student-Teacher Ratio and Interactions on Student/Teacher Performance in High School Scenarios

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Abstract

We develop a model that incorporates the impact of student-teacher ratio on the performance dynamics of both teachers and students. The model assumes that the members of both populations may be found in three dynamic states: positive, discouraged and reluctant. The role of complex nonlinear interactions between students and teachers, as well as the role of recruitment and intervention, are studied via analytic and numerical studies. Using center manifold theory we find conditions for the existence of a backward bifurcation that support endemic stationary states below the critical threshold value, $\mathcal{R}_0 < 1$, when normally only a positive environment would be supported. Our simulations show that in order to maintain a positive environment for students and teachers, \mathcal{R}_0 must be reduced significantly. Since \mathcal{R}_0 is a function of student-teacher ratio this can be achieved by decreasing class size.

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Background

Improving the quality of education is a challenge concerning many people in the United States. Many Americans feel the educational system should provide knowledge, information, and skills to compete in the world market. The educational system has yet to convince most Americans that students are comprehending the material presented in the curriculum and that students have the ability to be productive given the skills gained at school. A study by The National Center for Education Statistics (NCES 1999) emphasizes the need of a supportive work environment for teachers and students. This study focuses on class size, or student-teacher ratio, and attitudes of students and teachers in full time public schools because these two elements are not mutually exclusive.

While research on class size is controversial, some studies present the positive effects of reduced class size on student performance. The studies in favor of reducing class size (Project STAR, SAGE program, NAEP Central City Study, and CSR program in California) demonstrate how academic performance improves while also observing stronger teacher effectiveness in smaller classes. It seems reasonable to assume smaller class size facilitates teachers' work by reducing the number of disruptions and increasing the level of attention and participation per student (Achilles 1996). While class size is a factor, sometimes reducing class size does not significantly change the quality of the classroom environment. However, individual attitudes and motivation need to be considered because even talented people need stimulating and rewarding workplaces (Fullan with Stiegelbauer 1991). This is a unique characteristic of an educational environment like a public high school where both teachers' and students' attitudes have a significant role, since attitudes of both populations can influence the performance of both groups.

On October 21, 1998 a law was signed establishing The Federal Class Size Reduction program for 36 major city school systems (Council of the Great City Schools October 2000). For a class size reduction program to be successful, changes need to be focused on both teachers and students. Students in smaller classes receive more attention from the teacher allowing them to concentrate more in a smaller environment. Since teachers feel smaller classes are more manageable, and they spend less time disciplining the class, teachers are more likely to have positive attitudes in smaller classes. In turn, the teachers' improved motivation affects the attitudes and motivation of students and other teachers, improving their performance. Likewise, the students' improved attitudes positively affect the attitudes of other students and of teachers. This web of interactions may be the key of why the teachers and students achieve so much more in a smaller, enjoyable, and challenging setting.

Since educated individuals are a benefit to society, education is a major concern for government and related agencies. In response to this concern, many qualitative and quantitative studies examine different aspects that correlate to student performance such as teacher qualification, professional development, school resources and class size. Research shows that

there exists an important relationship between student-teacher ratio and student achievement (NCES 1999). This not only agrees with the intuitive concept that less students per teacher gives more opportunity for interactive and effective learning, but it also implies that class size has some effect on teachers and students. Consequently, quality of instruction can vary per classroom based on student-teacher ratio.

We choose to focus on the effect class size has on teachers and students manifested by their attitudes, since both students and teachers can become discouraged when working in stressful conditions. Studies show that the interactions between the population of students and teachers affects the teaching-learning process, however researchers have been unable to quantify the impact of these interactions. These contacts are difficult to interpret and quantify because of the subjective nature of assessing students' and teachers' motivation while they are in the school environment.

There is a wide range of student-teacher ratios throughout the United States and perceptions as to their effects on the educational process vary. Educators and policy makers need to know which class size is more efficient given the available resources, since it is almost impossible to have a teacher for only a few students. It is necessary to know how class size impacts student performance because changes in policies and regulations can be made to improve the quality of education.

Introduction

There are many factors that affect student achievement, but the purpose of this study is to explore and analyze the effects of student-teacher ratio as well as student-teacher interaction dynamics at the high school level. This study attempts to quantify and analyze the impact of student-teacher ratio on high school student performance based on the assumption that higher student-teacher ratios will tend to discourage teachers and students. We also consider how this discouragement effect will be increased by the interactions of the individuals within both groups and between both groups.

In our model, we employ an epidemic modeling approach to study the impact of student-teacher ratio and interactions among students and teachers on the performance of students and teachers in a high school setting. The populations of students and teachers in this model are characterized as positive, discouraged, or reluctant. These individuals are defined as those that are susceptible to the attitudes of the surrounding community.

This article is organized as follows: Section 1 introduces the three class model and describes the parameters; Section 2 introduces the two class model; Section 3 provides the analysis of the two class model; Section 4 provides the analysis of the three class model; Section 5 results; Section 6 provides the conclusions; and Section 7 outlines work in progress.

1 The Three Class Model

This model is given by a system of differential equations, which simultaneously describes the interactions among teachers and students as two separate populations and between teachers and students interacting in public high schools. We assume the population is uniform and homogeneously mixing, that is, there is no bias within the interactions of students and teachers. We classify the populations into six groups: positive teachers (P_1), discouraged teachers (D_1), reluctant teachers (R_1), positive students (P_2), discouraged students (D_2), and reluctant student (R_2).

In this model, we define positive teachers as those who are rated excellent by students and faculty. Discouraged teachers are those who are noticeably challenged by their environment, therefore affecting their performance in the classroom. Reluctant teachers are teachers who are rated poorly by students and faculty. We describe positive students as motivated and likely to achieve high scores, while discouraged students lack motivation and obtain lower scores. Reluctant students refers to the students who are poorly motivated and ranked lower in their class.

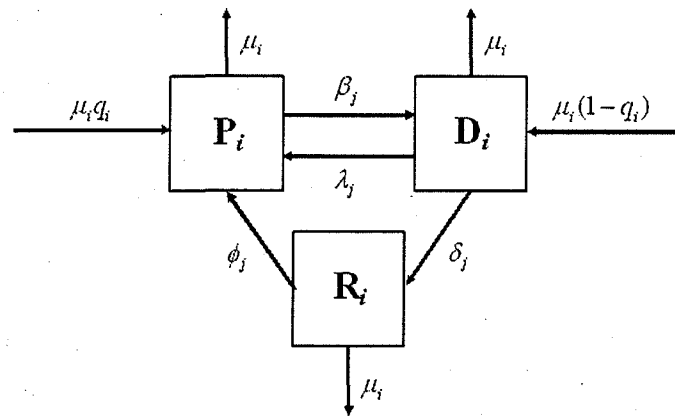


Figure 1: Three Class Model for Students and Teachers

The parameters in this model are described as follows:

- r is the student-teacher ratio.
- μ_1 is the teacher attrition and migration rate accounting for teachers leaving the profession, by retiring, quitting, being fired, or by changing schools.
- $\beta_1(r)$ is the teacher discouragement rate, a function of the student-teacher ratio.
- $\lambda_1(r)$ is the teacher encouragement rate, a function of the student-teacher ratio.

- δ_1 is the teacher reluctance rate, the rate at which discouraged teachers become reluctant due to interactions with reluctant students and/or teachers.
- ϕ_1 is the teacher miracle rate, the rate at which reluctant teachers become positive due to very close interactions with positive students and/or teachers.
- μ_2 is the student drop-out rate, the rate at which students leave high school without obtaining a secondary school credential or without enrolling in another educational program.
- $\beta_2(r)$ is the student discouragement rate, a function of the student-teacher ratio.
- $\lambda_2(r)$ is the student encouragement rate, a function of the student-teacher ratio.
- δ_2 is the student reluctance rate, the rate at which discouraged students become reluctant due to interactions with reluctant students and/or teachers.
- ϕ_2 is the student miracle rate, the rate at which reluctant students become positive due to very close interactions with positive students and/or teachers.

Note that only $\beta_1(r), \beta_2(r), \lambda_1(r), \lambda_2(r)$ are functions of the student-teacher ratio, because we assume these are more sensitive to class size. For the analysis we shall denote them simply as $\beta_1, \beta_2, \lambda_1,$ and λ_2 . The β_i functions are increasing functions. In other words, as the student-teacher ratio increases, so do the discouragement rates for the students and teachers. Moreover, λ_i are exponentially decreasing functions, so that as the student-teacher ratio decreases, the encouragement rate increases.

We assume that recruitment into the positive population occurs at a rate $q_i \mu_i N_i$ where q_i is the proportion of the population entering the respective positive class. Likewise, recruitment into the discouraged population occurs at a rate $(1 - q_i) \mu_i N_i$. Also, we assume that no individuals enter directly into the reluctant individual class. The range of q_i is between zero and one ($0 \leq q_i \leq 1$), where 0 implies that all individuals entering the population are discouraged and 1 implies that all recruited individuals are positive.

Using Fig. 1 we can formulate the model as follows:

$$\begin{aligned}
\dot{P}_1 &= \mu_1 q_1 N_1 + \lambda_1(r) \frac{P_1}{N_1} D_1 + \lambda_2(r) \frac{P_2}{N_2} D_1 + \phi_1 \frac{P_1}{N_1} R_1 + \phi_2 \frac{P_2}{N_2} R_1 \\
&\quad - \beta_1(r) P_1 \frac{D_1 + R_1}{N_1} - \beta_2(r) P_1 \frac{D_2 + R_2}{N_2} - \mu_1 P_1 \\
\dot{D}_1 &= \mu_1 (1 - q_1) N_1 + \beta_1(r) P_1 \frac{D_1 + R_1}{N_1} + \beta_2(r) P_1 \frac{D_2 + R_2}{N_2} \\
&\quad - \lambda_1(r) \frac{P_1}{N_1} D_1 - \lambda_2(r) \frac{P_2}{N_2} D_1 - \delta_1 \frac{R_1}{N_1} D_1 - \delta_2 \frac{R_2}{N_2} D_1 - \mu_1 D_1 \\
\dot{R}_1 &= \delta_1 \frac{R_1}{N_1} D_1 + \delta_2 \frac{R_2}{N_2} D_1 - \phi_1 \frac{P_1}{N_1} R_1 - \phi_2 \frac{P_2}{N_2} R_1 - \mu_1 R_1 \\
\dot{P}_2 &= \mu_2 q_2 N_2 + \lambda_1(r) \frac{P_1}{N_1} D_2 + \lambda_2(r) \frac{P_2}{N_2} D_2 + \phi_1 \frac{P_1}{N_1} R_2 + \phi_2 \frac{P_2}{N_2} R_2 \\
&\quad - \beta_1(r) P_2 \frac{D_1 + R_1}{N_1} - \beta_2(r) P_2 \frac{D_2 + R_2}{N_2} - \mu_1 P_2 \\
\dot{D}_2 &= \mu_2 (1 - q_2) N_2 + \beta_1(r) P_2 \frac{D_1 + R_1}{N_1} + \beta_2(r) P_2 \frac{D_2 + R_2}{N_2} \\
&\quad - \lambda_1(r) \frac{P_1}{N_1} D_2 - \lambda_2(r) \frac{P_2}{N_2} D_2 - \delta_1 \frac{R_1}{N_1} D_2 - \delta_2 \frac{R_2}{N_2} D_2 - \mu_2 D_2 \\
\dot{R}_2 &= \delta_1 \frac{R_1}{N_1} D_2 + \delta_2 \frac{R_2}{N_2} D_2 - \phi_1 \frac{P_1}{N_1} R_2 - \phi_2 \frac{P_2}{N_2} R_2 - \mu_1 R_2 \\
N_1 &= P_1 + D_1 + R_1 \\
N_2 &= P_2 + D_2 + R_2
\end{aligned} \tag{1}$$

The terms in the above equations for the teacher population can be interpreted as follows:

- $\mu_1 q_1 N_1$ is the recruitment rate of teachers into the positive class.
- $\lambda_1 \frac{P_1}{N_1} D_1$ is the proportion of discouraged teachers that become positive after interacting with positive teachers.
- $\lambda_2 \frac{P_2}{N_2} D_1$ is the proportion of discouraged teachers that become positive after interacting with positive students.
- $\phi_1 \frac{P_1}{N_1} R_1$ is the proportion of reluctant teachers that become positive after interacting with positive teachers.
- $\phi_2 \frac{P_2}{N_2} R_1$ is the proportion of reluctant teachers that become positive after interacting with positive students.

- $\beta_1 P_1 \frac{D_1 + R_1}{N_1}$ is the proportion of positive teachers that become discouraged after interacting with discouraged and reluctant teachers.
- $\beta_2 P_1 \frac{D_2 + R_2}{N_2}$ is the proportion of positive teachers that become discouraged after interacting with discouraged and reluctant students.
- $\mu_1 P_1$ the rate of positive teachers leaving the positive class.
- $\mu_1(1 - q_1)N_1$ is the recruitment rate of teachers into the discouraged class.
- $\delta_1 \frac{R_1}{N_1} D_1$ is the proportion of discouraged teachers that become reluctant after interacting with reluctant teachers.
- $\delta_2 \frac{R_2}{N_2} D_1$ is the proportion of discouraged teachers that become reluctant after interacting with reluctant students.
- $\mu_1 D_1$ is the rate of discouraged teachers leaving the discouraged class.
- $\mu_1 R_1$ is the rate of reluctant teachers leaving the reluctant class.

Since the form of the equations for the student model are symmetrical with respect to the subscripts, the terms in the student equations can be interpreted similarly.

By adding the equations for the population of teachers we have the relationship,

$$\dot{N}_1 = \dot{P}_1 + \dot{D}_1 + \dot{R}_1.$$

Assuming the population has reached a steady state, we can let $N_1 = 1$ so that $\dot{N}_1 = 0$, implying that the population of teachers is constant. Similarly, we can show that the population of students is also constant. Thus, without loss of generality, we can consider $N_1 = 1$ and $N_2 = 1$ so that (P_1) , (D_1) , (R_1) , (P_2) , (D_2) , and (R_2) will represent proportions, all having values between zero and one.

We also note that now, since $N_1 = P_1 + D_1 + R_1 = 1$ and $N_2 = P_2 + D_2 + R_2 = 1$, we can make the substitutions $R_1 = 1 - P_1 - D_1$ and $R_2 = 1 - P_2 - D_2$. This reduces the system of differential equations from six dimensions to four dimensions as follows.

$$\begin{aligned} \dot{P}_1 &= \mu_1 q_1 + \lambda_1(r) D_1 P_1 + \lambda_2(r) D_1 P_2 + \phi_1(1 - P_1 - D_1) P_1 + \phi_2(1 - P_1 - D_1) P_2 \\ &\quad - \beta_1(r) P_1(1 - P_1) - \beta_2(r) P_1(1 - P_2) - \mu_1 P_1 \\ \dot{D}_1 &= \mu_1(1 - q_1) + \beta_1(r) P_1(1 - P_1) + \beta_2(r) P_1(1 - P_2) \\ &\quad - \lambda_1(r) D_1 P_1 - \lambda_2(r) D_1 P_2 - \delta_1 D_1(1 - P_1 - D_1) - \delta_2 D_1(1 - P_2 - D_2) - \mu_1 D_1 \\ \dot{P}_2 &= \mu_2 q_2 + \lambda_1(r) D_2 P_1 + \lambda_2(r) D_2 P_2 + \phi_1(1 - P_2 - D_2) P_1 + \phi_2(1 - P_2 - D_2) P_2 \\ &\quad - \beta_1(r) P_2(1 - P_1) - \beta_2(r) P_2(1 - P_2) - \mu_2 P_2 \\ \dot{D}_2 &= \mu_2(1 - q_2) + \beta_1(r) P_2(1 - P_1) + \beta_2(r) P_2(1 - P_2) \\ &\quad - \lambda_1(r) D_2 P_1 - \lambda_2(r) D_2 P_2 - \delta_1 D_2(1 - P_1 - D_1) - \delta_2 D_2(1 - P_2 - D_2) - \mu_2 D_2. \end{aligned}$$

1.1 Analysis of the Three Class Model

Because of the complexity of the three class model, before analyzing it we consider a two class model that no longer takes into account the effects of the reluctant class of teachers or students. This new two class model allows us to develop a preliminary understanding of the dynamics of the three class model.

2 The Two Class Model

The two class model is portrayed by the following diagram and the parameters are defined as in the three class model. Using Fig.2 and recalling that we can assume that the system

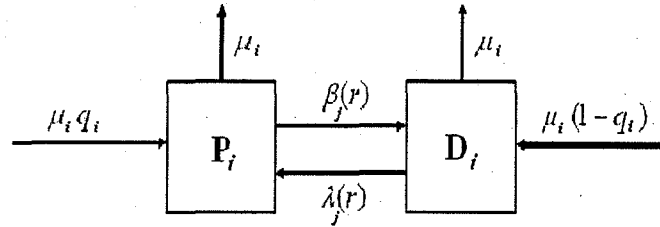


Figure 2: Two Class Model for Students and Teachers

has reached a steady state so that we may let $N_1 = 1$ and $N_2 = 1$, without loss of generality, we formulate the following model:

$$\begin{aligned} \dot{P}_1 &= \mu_1 q_1 + \lambda_1 P_1 D_1 + \lambda_2 P_2 D_1 - \beta_1 D_1 P_1 - \beta_2(r) D_2 P_1 - \mu_1 P_1 \\ \dot{D}_1 &= \mu_1(1 - q_1) + \beta_1 D_1 P_1 + \beta_2 D_2 P_1 - \lambda_1 P_1 D_1 - \lambda_2 P_2 D_1 - \mu_1 D_1 \\ \dot{P}_2 &= \mu_2 q_2 + \lambda_1 P_1 D_2 + \lambda_2 P_2 D_2 - \beta_1 D_1 P_2 - \beta_2 D_2 P_2 - \mu_2 P_2 \\ \dot{D}_2 &= \mu_2(1 - q_2) + \beta_1 D_1 P_2 + \beta_2 D_2 P_2 - \lambda_1 P_1 D_2 - \lambda_2 P_2 D_2 - \mu_2 D_2 \end{aligned}$$

The terms in the above equations are the same as they were defined in the three class model. Also, since the population is constant, as with the three class model, we can reduce the system of differential equations from four dimensions to two dimensions with the substitutions $D_1 =$

$1 - P_1$ and $D_2 = 1 - P_2$.

$$\begin{aligned} \dot{P}_1 &= \mu_1 q_1 + \lambda_1 P_1(1 - P_1) + \lambda_2 P_2(1 - P_1) \\ &\quad - \beta_1(1 - P_1)P_1 - \beta_2(1 - P_2)P_1 - \mu_1 P_1 \\ N_1 &= P_1 + D_1 = 1 \\ \dot{P}_2 &= \mu_2 q_2 + \lambda_1 P_1(1 - P_2) + \lambda_2 P_2(1 - P_2) \\ &\quad - \beta_1(1 - P_1)P_2 - \beta_2(1 - P_2)P_2 - \mu_2 P_2 \\ N_2 &= P_2 + D_2 = 1. \end{aligned}$$

3 Stability of Discouragement Free Equilibrium Point for Two Class Model

We begin the analysis of the two class model by considering the case where there are only positive teachers and students in the system after the system has reached a steady state. For the discouragement free equilibrium point to exist, where there are only people in the positive classes, we can only have recruitment into the positive classes. Mathematically, this means that we are restricting the model to the special case when $q_1 = 1$ and $q_2 = 2$. The stability conditions are based on the trace and the determinant of the Jacobian matrix for the two class model evaluated at the discouragement free equilibrium point.

First we compute the general Jacobian matrix for the two class model at $X_1^* = (P_1^*, P_2^*)$:

$J_{P_1^*, P_2^*} =$

$$\begin{bmatrix} (\lambda_1 - \beta_1)(1 - 2P_1^*) - (\lambda_2 - \beta_2)P_2^* - \beta_2 - \mu_1 & \beta_2 P_1^* \\ \lambda_1 - \lambda_1 P_2^* + \beta_1 P_2^* & (\lambda_2 - \beta_2)(1 - 2P_2^*) - (\lambda_1 - \beta_1)P_1^* - \beta_1 - \mu_2 \end{bmatrix}$$

Next we compute the Jacobian at the discouragement free equilibrium point for the two class model, $X_1 = (1, 1)$, which is the trivial case when all students and teachers are positive.

$$J_{1,1} = \begin{bmatrix} -\lambda_1 - \lambda_2 + \beta_1 - \mu_1 & \beta_2 \\ \beta_1 & -\lambda_1 - \lambda_2 + \beta_2 - \mu_2 \end{bmatrix}$$

The trace is negative when

$$\frac{\beta_1 + \beta_2}{2\lambda_1 + 2\lambda_2 + \mu_1 + \mu_2} < 1$$

and the determinant is positive when

$$\frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} < 1.$$

But, as in the case with the stability of the discouragement free equilibrium point for the three class model, for the two class model we have that

$$\frac{\beta_1 + \beta_2}{2\lambda_1 + 2\lambda_2 + \mu_1 + \mu_2} < 1 \text{ whenever } \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} < 1.$$

Thus, the discouragement free equilibrium point for the two class model is locally asymptotically stable when the condition for the determinant to be positive is satisfied, and unstable otherwise.

3.1 The Basic Discouragement Number for the Two Class Model

In order to compute the basic discouragement number, \mathfrak{R}_0 , for the two class model, we consider the approach proposed by Van den Driessche and Watmough (Driessche 2002). We begin by distinguishing the new generated cases of discouraged and reluctant individuals from all changes in the population.

Let $F(x)$ be the rate of appearance of new discouraged and reluctant individuals in compartment i . V is the transfer rate of individuals into compartment i by all other means, and out of compartment i . To determine the outcome of a “typical” discouraged individual introduced into the population, we study the dynamics of the linearized system for the two class model.

We start out with the equations for the discouraged classes.

$$\begin{aligned}\dot{D}_1 &= \mu_1(1 - q_1) + \beta_1 P_1 D_1 + \beta_2 P_1 D_2 - \lambda_1 D_1 P_1 - \lambda_2 D_1 P_2 - \mu_1 D_1 \\ \dot{D}_2 &= \mu_2(1 - q_2) + \beta_1 P_2 D_1 + \beta_2 P_2 D_2 - \lambda_1 D_2 P_1 - \lambda_2 D_2 P_2 - \mu_2 D_2\end{aligned}$$

The generalized system can be written as

$$\dot{x} = (F - V)(x)$$

The basic discouragement number for the two class model is determined using the second generation approach originally introduced by Diekmann *et al.*, [5].

$$\begin{aligned}\tilde{F} &= \begin{pmatrix} \dot{D}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} \beta_1 P_1^* D_1^* + \beta_2 P_1^* D_2^* \\ \beta_1 P_2^* D_1^* + \beta_2 P_2^* D_2^* \end{pmatrix} \\ \tilde{V} &= \begin{pmatrix} \dot{D}_1 \\ \dot{D}_2 \end{pmatrix} = \begin{pmatrix} -\mu_1(1 - q_1) + \lambda_1 D_1^* P_1^* + \lambda_2 D_1^* P_2^* + \mu_1 D_1^* \\ -\mu_2(1 - q_2) + \lambda_1 D_2^* P_1^* + \lambda_2 D_2^* P_2^* + \mu_2 D_2^* \end{pmatrix}\end{aligned}$$

For the discouragement free equilibrium, $X^* = (P_1^*, D_1^*, P_2^*, D_2^*) = (1, 0, 1, 0)$;

$$\begin{aligned} F &= \frac{\partial \tilde{F}}{\partial D_i}(X^*) = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \end{pmatrix}, \\ V &= \frac{\partial \tilde{V}}{\partial D_i}(X^*) = \begin{pmatrix} \lambda_1 + \lambda_2 + \mu_1 & 0 \\ 0 & \lambda_1 + \lambda_2 + \mu_2 \end{pmatrix}, \\ V^{-1} &= \begin{pmatrix} \frac{1}{\lambda_1 + \lambda_2 + \mu_1} & 0 \\ 0 & \frac{1}{\lambda_1 + \lambda_2 + \mu_2} \end{pmatrix} \end{aligned}$$

and

$$FV^{-1} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1 + \lambda_2 + \mu_1} & 0 \\ 0 & \frac{1}{\lambda_1 + \lambda_2 + \mu_2} \end{pmatrix} = \begin{pmatrix} \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \\ \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \end{pmatrix}.$$

We call FV^{-1} the next generation matrix for the model and set the basic discouragement number equal to the dominant eigenvalue. Thus,

$$\mathfrak{R}_0 = \rho(FV^{-1}) = \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2}$$

We call \mathfrak{R}_0 the basic discouragement rate. It is the number of positive students and teachers one discouraged student or teacher discourages on average throughout the discouraged individual's time in the system when introduced into a mostly positive environment. Since the model is coupled and discouraged individuals from one group affect positive individuals in the other group as well as in their own group, \mathfrak{R}_0 has to reflect the interactions between the two groups. That is why the basic discouragement number for the coupled system is the sum of the two terms which on their own represent the discouragement numbers for the respective teacher and student systems. The student-teacher interactions are also described in \mathfrak{R}_0 by the λ_i terms, the rates at which discouraged individuals become positive by having contact with positive individuals from either group. This explains the presence of the λ_2 term, the rate at which discouraged students become positive students, in the denominator of what would constitute the discouragement number for the teacher system that would otherwise only have terms reflecting teacher interactions, and vice versa.

Recalling that since β_i and λ_i are all functions of r , the student teacher ratio, we can consider the behavior of $\mathfrak{R}_0(r)$, see figure 3.

3.2 Existence of Endemic Equilibria for the Two Class Model

Now that we have established the condition for the stability of the discouragement free equilibrium point, we are interested in the existence of equilibria that include discouraged

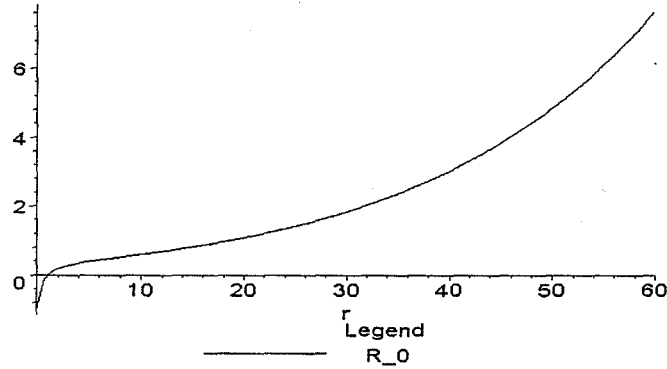


Figure 3: Re_0 as a function of student-teacher ratio

individuals, i.e. equilibria with $D_1 > 0$, $D_2 > 0$ when $\mathfrak{R}_0 > 1$. Although we were able to analyze the stability of the trivial equilibrium point, the complexity of the model has prevented us from analytically identifying the endemic equilibrium point(s). However, we can prove that there is an endemic equilibrium point when the discouragement free equilibrium point is unstable, that is when $\mathfrak{R}_0 > 1$.

Theorem:

There exists a positive endemic equilibria if $\mathfrak{R}_0 > 1$. Let \dot{P}_1 and \dot{P}_2 be equal to zero, that is

$$\mu_1 q_1 + (1 - P_1)[\lambda_1(r)P_1 + \lambda_2(r)P_2 - \beta_1 P_1] - \beta_2 P_1(1 - P_2) - \mu_1 P_1 = 0 \quad (2)$$

$$\mu_2 q_2 + (1 - P_2)[\lambda_1(r)P_1 + \lambda_2(r)P_2 - \beta_2 P_2] - \beta_1 P_2(1 - P_1) - \mu_2 P_2 = 0 \quad (3)$$

By solving in equation (2) P_2 in terms of P_1 we get,

$$P_2 = f(P_1) = \frac{\mu_1 q_1 + P_1(1 - P_1)(\lambda_1 - \beta_1) - P_1(\mu_1 + \beta_2)}{P_1(\lambda_2 - \beta_2) - \lambda_2}$$

Now, substituting $f(P_1)$ into (3) yields,

$$F(P_1) = [1 - f(P_1)][f(P_1)(\lambda_2 - \beta_2) + \lambda_1 P_1] - \beta_1 f_1(P_1)(1 - P_1) + \mu_2 [q_2 - f(P_1)]$$

Let $F(P_1)$ be a continuous function on the closed interval $[0,1]$, such that $F(0) > 0$ and $F(1) < 0$. It follows from the Intermediate Value Theorem that there is a real number $c \in [0,1]$ with $F(c) = 0$. So, we proceed by finding $F(0)$ and $F(1)$ and their signs at these two points, thus we get the following expression,

$$\begin{aligned} F(0) &= [1 - f(0)][f(0)(\lambda_2 - \beta_2)] + \mu_2 [q_2 - f(0)] \\ F(1) &= [1 - f(1)][f(1)(\lambda_2 - \beta_2) + \lambda_1] + \mu_2 [q_2 - f(1)] \end{aligned}$$

Since its proved that when $P_1 = 1 \leftrightarrow P_2 = 1$, and $P_1 = 0 \leftrightarrow P_2 = 0$;

$$f(0) = 0 \text{ and } f(1) = 1 \quad (4)$$

Then, by substituting (4) and using hypothesis (??), we get the following result,

$$\begin{aligned} F(0) &= \mu_2 q_2 > 0 \\ F(1) &= \mu_2 q_2 - 1 < 0 \end{aligned}$$

Therefore, there exists a positive endemic equilibrium point $X_l = (P_1^*, P_2^*)$, such that $0 < P_1 < 1$ and $0 < P_2 < 1$ when $\mathfrak{R}_0 > 1$. \square

3.3 Behavior of Discouragement and Encouragement Functions

This are the function that are used in the numerical analysis. They connect the project to the numerical analysis.

In order to further understand the role of the student-teacher ratio (r), we explore specific functions for $\lambda(r)$ and $\beta(r)$. The rate at which positive individuals go into the discouraged class increases as r increases. This suggests that as the number of students assigned to each teacher increases, the likelihood that students will become discouraged will be higher. To describe the dynamics of discouraged individuals becoming positive we consider the function $\lambda(r)$. This function describes an almost exponential decrease in growth of discouraged individuals leaving the discouraged class as they become positive. In this case, as the student-teacher ratio increases, less discouraged students become positive.

We propose the following two functions to describe the dynamics of the rates at which individuals become discouraged and positive, respectively. In order to determined what functions we were going to use to simulate the behavior of the rates, we take into consideration the following conditions: $\beta(1) = 0$, $\beta''(r) > 0$ when $1 < r < r_c$, $\beta''(r_c) = 0$ when $\beta''(r) < 0$ and $r_c < r$, and finally $\lambda''(r) > 0$, where r_c is the critical student-teacher ratio describing the switch from having a majority of discouraged and reluctant attitudes to having a majority of encouraged attitudes.

$$\begin{aligned} \beta_1(r) &= (r - 1)/(1 + 3r) \\ \beta_2(r) &= (r - 1)/(1 + 2r) \\ \lambda_1(r) &= e^{-0.06r} \\ \lambda_2(r) &= e^{-0.05r} \end{aligned}$$

In order to study the impact of $\beta_i(r)$ and $\lambda_i(r)$ for $i=1,2$ on \mathfrak{R}_0 , we explore three separate regions of r for the proposed functions applied to the student and teacher model. The behavior of \mathfrak{R}_0 is determined by studying regions I, II, and at $\lambda(r) = \beta(r)$ (see Figures 4 and 5). Region I describes the dynamics under which $\beta(r)$ is smaller than $\lambda(r)$ and region II when $\lambda(r)$ is smaller than $\beta(r)$.

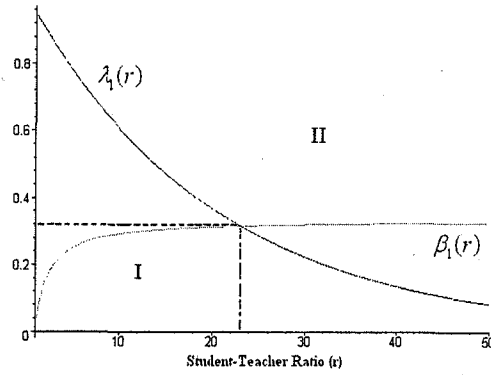


Figure 4: Comparison of $\beta_1(r)$ and $\lambda_1(r)$ for teachers

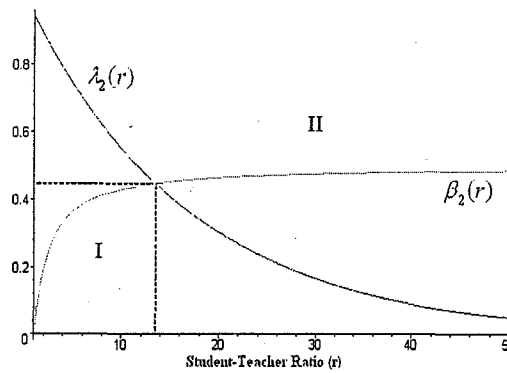


Figure 5: Comparison of $\beta_2(r)$ and $\lambda_2(r)$ for students

3.4 Uncertainty and Sensitivity Analysis of \mathcal{R}_0

The basic discouragement number (\mathcal{R}_0) describes the invasion of a disease (ideology, attitude, etc.), and in this model of discouragement, in a population. To explore the sensitivity of \mathcal{R}_0 to the variability of the parameters in our proposed models, we let ξ represent any of the eight parameters ($\beta_i, \lambda_i, \mu_i, q_i$ for $i = 1, 2$ and $i \neq j$). Consider a small perturbation to ξ by $\Delta\xi$. A perturbation in ξ suggests that a perturbation will affect \mathcal{R}_0 ($\Delta\mathcal{R}_0$) as well. The normalized sensitivity index \mathcal{S}_ξ is the ratio of the corresponding normalized changes [3]. We define the sensitivity index for parameter ξ

$$\mathcal{S}_\xi := \frac{\Delta\mathcal{R}_0}{\mathcal{R}_0} / \frac{\Delta\xi}{\xi} = \frac{\xi}{\mathcal{R}_0} \frac{\partial\mathcal{R}_0}{\partial\xi}$$

We calculate the indices \mathcal{S}_ξ for the parameters in our model. In section 3, we calculated the equilibrium in the absence of discouraged and reluctant individuals, and the basic reproductive number:

$$\mathcal{R}_0 = \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2}$$

Considering \mathfrak{R}_0 and calculating the sensitivity for each of its eight parameters, we find the following normalized sensitivity indices.

$$\begin{aligned} S_{\beta_1} &= \frac{\beta_1}{\zeta_1 \mathfrak{R}_0} \\ S_{\beta_2} &= \frac{\beta_2}{\zeta_2 \mathfrak{R}_0} \\ S_{\lambda_1} &= -\frac{\lambda_1}{\mathfrak{R}_0} \left(\frac{\beta_1}{\zeta_1^2} + \frac{\beta_2}{\zeta_2^2} \right) \\ S_{\lambda_2} &= -\frac{\lambda_2}{\mathfrak{R}_0} \left(\frac{\beta_1}{\zeta_1^2} + \frac{\beta_2}{\zeta_2^2} \right) \\ S_{\mu_1} &= -\frac{\beta_1 \mu_1}{\zeta_1^2 \mathfrak{R}_0} \\ S_{\mu_2} &= -\frac{\beta_2 \mu_2}{\zeta_2^2 \mathfrak{R}_0} \end{aligned}$$

where $\zeta_1 = \lambda_1 + \lambda_2 + \mu_1$ and $\zeta_2 = \lambda_1 + \lambda_2 + \mu_2$.

From the sensitivity indices provided in (3.4), it can be observed that S_{β_1} and S_{β_2} are always positive. In contrast, indices S_{λ_1} , S_{λ_2} , S_{μ_1} , and S_{μ_2} are always negative. From (3.4), it is clear that all indices are functions of the parameters, hence the values of the indices depend on the particular values chosen for each parameter. Furthermore, it can be observed that

$$\begin{aligned} S_{\mu_1} &= \frac{-\mu_1}{\zeta_1} S_{\beta_1} \\ S_{\mu_2} &= \frac{-\mu_2}{\zeta_2} S_{\beta_2} \\ S_{\lambda_1} &= -\lambda_1 \left(\frac{S_{\beta_1}}{\zeta_1} + \frac{S_{\beta_2}}{\zeta_2} \right) \\ S_{\lambda_2} &= -\lambda_2 \left(\frac{S_{\beta_1}}{\zeta_1} + \frac{S_{\beta_2}}{\zeta_2} \right). \end{aligned}$$

From the above results it can also be observed that

$$S_{\beta} > S_{\lambda}.$$

| Positive Sensitivity Indices | | Negative Sensitivity Indices | |
|------------------------------|---------|------------------------------|--------|
| $S_{\beta_1} = 0.4041$ | -2.47 % | $S_{\lambda_1} = -0.4354$ | 2.30 % |
| $S_{\beta_2} = 0.5959$ | -1.68 % | $S_{\lambda_2} = -0.5224$ | 1.91 % |
| | | $S_{\mu_1} = -0.04004$ | 25.0 % |
| | | $S_{\mu_2} = -0.002159$ | 100 % |

Table 1. Sensitivity indices for \mathfrak{R}_0 .

From the above table we can see that the positive sensitivity indices are β_1 and β_2 and the negative sensitivity indices are λ_1 , λ_2 , μ_1 and μ_2 . A positive sensitivity index means that as the parameter value increases, the value of \mathfrak{R}_0 increases while a negative sensitivity index means that as the parameter increases, the value of \mathfrak{R}_0 decreases. The sensitivity analysis indicates that \mathfrak{R}_0 is most sensitive to β_2 , the discouragement rate for students, and λ_2 , the encouragement rate for students.

3.5 Center Manifold Theory

In order to perform stability analysis on the endemic equilibrium points, we apply center manifold theory for epidemic models and review the normal forms for bifurcations (Kribs-Zaleta 1998). This way we can perform bifurcation analysis without having explicit analytic solutions for the endemic equilibrium points. Center manifold theory concentrates on a subspace of the original state space for the model where the eigenvalues of the system whose real part crosses zero is usually less than the dimension of the system. This center manifold is an attractor in the state space, so it allows us to consider the dynamics on the center manifold in order to understand the dynamics of the system in terms of stability of the equilibria. (Kribs-Zaleta 1998)

The first step is to consider the two class model with parameter β_1^* and equilibrium value $\dot{x} = f(x_1, x_2, \Phi)$. Then to translate the system so that the bifurcation point is at the origin we let

$$f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \text{ and } f \in \mathcal{C}^2(\mathbb{R}^2 \times \mathbb{R}),$$

$$f(x_1, x_2, \Phi) \equiv 0 \text{ for all } \Phi.$$

Next we choose β_1 as the bifurcation parameter so that we can make a linear transformation. $\mathfrak{R}_0(\beta_1^*) = 1$ that is $\beta_1 = \beta_1^* \Leftrightarrow \mathfrak{R}_0 = 1$, where

$$\beta_1^* = \left(1 - \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \right) (\lambda_1 + \lambda_2 + \mu_1)$$

The Jacobian of the translated system about $P_1 = \tilde{P}_1 + 1$ is $P_2 = \tilde{P}_2 + 1$ and $\beta_1 = \Phi + \beta_1^*$

$$\begin{pmatrix} (2\tilde{P}_1 + 1)(\Phi + \beta_1^* - \lambda_1) + \tilde{P}_2(\beta_2 - \lambda_2) - \lambda_2 - \mu_1 & \tilde{P}_1(\beta_2 - \lambda_2) + \beta_2 \\ \tilde{P}_2(\Phi + \beta_1^* - \lambda_1) + (\Phi + \beta_1^*) & (2\tilde{x}_2 + 1)(\beta_2 - \lambda_2) + \tilde{x}_1(\Phi + \beta_1^* - \lambda_1) - \lambda_1 - \mu_2 \end{pmatrix}.$$

Then we compute the Jacobian matrix for the translated system around the disease-free equilibrium $(0, 0)$ with Φ evaluated at 0,

$$J_{(0,0,0)} = \begin{pmatrix} \beta_1^* - (\lambda_1 + \lambda_2 + \mu_1) & \beta_2 \\ \beta_1^* & \beta_2 - (\lambda_1 + \lambda_2 + \mu_2) \end{pmatrix}.$$

We calculate the right and left dominant eigenvector ω and ν corresponding to the dominant eigenvalue $\lambda = 0$.

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \mu_2 \\ \lambda_1 + \lambda_2 + \mu_1 \end{pmatrix}$$

$$\nu = (\nu_1 \ \nu_2) = (\lambda_1 + \lambda_2 + \mu_2 \ \lambda_1 + \lambda_2 + \mu_1).$$

Let f_k be the k th component of f and

$$a = \sum_{k,i,j=1}^n \nu_k \omega_i \omega_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0, 0),$$

$$b = \sum_{k,i=1}^n \nu_k \omega_i \frac{\partial^2 f_k}{\partial x_i \partial \Phi}(0, 0, 0).$$

The local dynamics of the system around $(0, 0, 0)$ are totally determined by a and b where

$$a = (\lambda_1 + \lambda_2 + \mu_2) \left(\left(1 - \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \right) (\lambda_1 + \lambda_2 + \mu_1) - \lambda_1 \right) + (\beta_2 - \lambda_2)(\lambda_1 + \lambda_2 + \mu_1)[(\lambda_1 + \lambda_2 + \mu_2)^2 + (\lambda_1 + \lambda_2 + \mu_1)^2],$$

and

$$b = (\lambda_1 + \lambda_2 + \mu_2)^2 + (\lambda_1 + \lambda_2 + \mu_1)(\lambda_1 + \lambda_2 + \mu_2).$$

For there to be a backward bifurcation we need $a > 0$ (Song 2003). By introducing specific functions for $\beta(r)$ and $\lambda(r)$ that depend on the student-teacher ratio, we conclude that a backward bifurcation occurs as long as

$$1 > \frac{\lambda_1(r)}{\lambda_1(r) + \lambda_2(r) + \mu_1} + \frac{\lambda_2(r)}{\lambda_1(r) + \lambda_2(r) + \mu_2} = C(r)$$

at $\mathfrak{R}_0 = 1$. Since λ_1 and λ_2 are functions of the student-teacher ratio, $C(r)$ is in terms of the class size as well. Figure(6) shows the behavior of the $C(r)$ as we vary the student-teacher ratio. To confirm the existence of a backward bifurcation when $\mathfrak{R}_0 < 1$, we looked at the

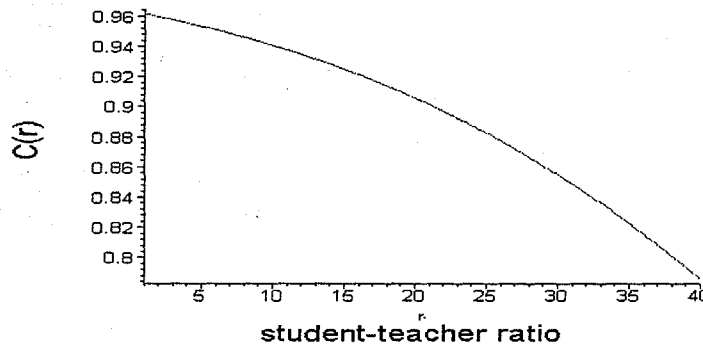


Figure 6: Condition for Backward Bifurcation

basin of attraction when $C(r) < 1$ and $\mathfrak{R}_0 < 1$. Figure 7 shows that there are two attractors, one which would be the locally asymptotically stable discouragement free equilibrium point and the second would be a locally asymptotically stable endemic equilibrium point.

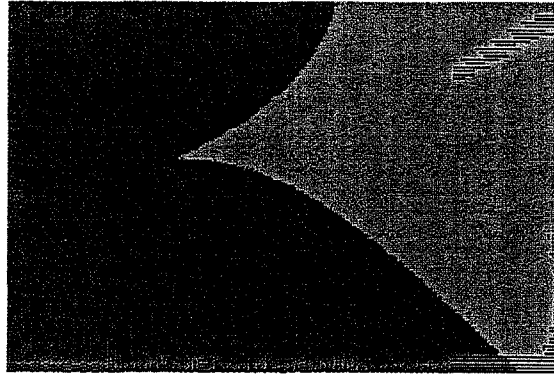


Figure 7: Two attractors

3.6 Numerical Simulations of the Two Class Model

| Teacher Parameters | Estimation | Student Parameters | Estimation |
|--------------------|--------------------|--------------------|--------------------|
| P_1 | 69 | P_2 | 500 |
| D_1 | 31 | D_2 | 500 |
| β_1 | $\frac{r-1}{1+3r}$ | β_2 | $\frac{r-1}{1+2r}$ |
| λ_1 | $e^{-0.6r}$ | λ_2 | $e^{-0.5r}$ |
| μ_1 | 0.121 | μ_2 | 0.004 |
| q_1 | 0.7 | q_2 | 0.5 |

Table 1: Initial Conditions

Using the initial conditions we obtained from our data (see Table 1) and by looking at a student-teacher ratio (r) of 18, gives the following graph: In this numerical simulation, figure 8, the dynamics reach a stable endemic equilibrium point even though \mathcal{R}_0 is less than one because our data is within the range of a backward bifurcation. This indicates that at $r=18$, the proportions of positive students and teachers remain greater than those of discouraged students and teachers.

By increasing the student-teacher ratio to 19 (see Fig 9), the dynamics of the system are switched; the proportion of positive students and teachers are no longer greater than that of those who are discouraged.

In figure 10, you will notice that both positive and negative teachers exist. For this to happen with $\mathcal{R}_0 > 1$, an endemic equilibrium point outside of the backward bifurcation must exist.

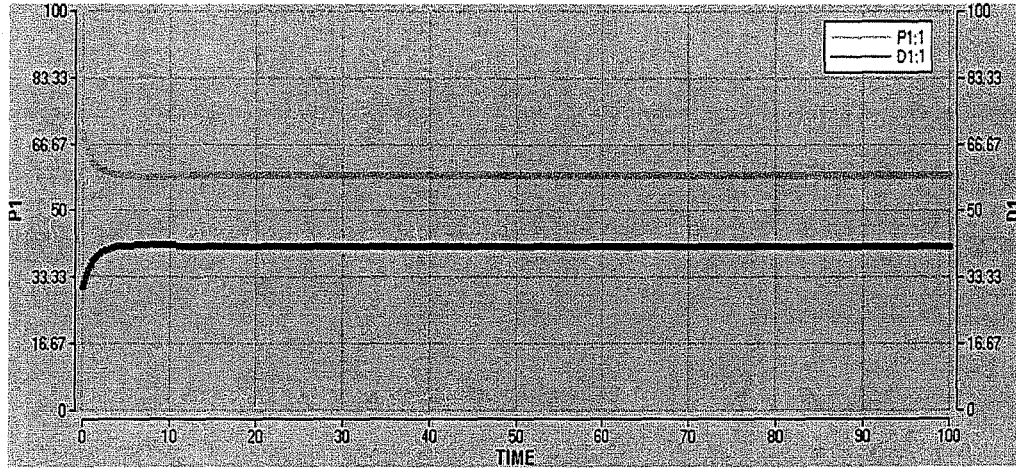


Figure 8: $r=18, \mathfrak{R}_0=0.89$

4 Discouragement Free Equilibrium and Stability for the Three Class Model

Now that we have analyzed the two call model, we are ready to continue with the analysis of the three class model. We begin the analysis of the three class model by considering the case where there are only positive teachers and students in the system after the system has reached a steady state. We analyze the stability of the equilibrium point by linearizing around it and checking the stability conditions based on the trace and the determinant of the Jacobian matrix evaluated at the discouragement free equilibrium point.

First we compute the general Jacobian matrix at $X_1^* = (P_1^*, D_1^*, P_2^*, D_2^*)$:

$J_{P_1^*, D_1^*, P_2^*, D_2^*} =$

$$\begin{bmatrix} \lambda_1 D_1^* + \phi_1(1 - P_1^* - D_1^*) - \beta_2 + \beta_2 P_2^* - \beta_1 + 2\beta_1 P_1^* - \mu_1 & \lambda_1 P_1^* + \lambda_2 P_2^* & \dots & \dots \\ \beta_1 - 2\beta_1 P_1^* + \beta_2 - \beta_2 P_2^* - \lambda_1 D_1^* & -\lambda_1 P_1^* - \lambda_2 P_2^* - \delta_1(1 - P_1^* - D_1^*) - \delta_2(1 - P_2^* - D_2^*) - \mu_1 & \dots & \dots \\ \lambda_1 D_2^* + \phi_1(1 - P_2^* - D_2^*) + \beta_1 P_2^* & 0 & \dots & \dots \\ -\beta_1 P_2^* - \lambda_1 D_2^* & 0 & \dots & \dots \\ \dots & \lambda_2 D_1^* + \phi_2(1 - P_1^* - D_1^*) + \beta_2 P_1^* & 0 & \dots \\ \dots & -\beta_2 P_1^* - \lambda_2 D_1^* & 0 & \dots \\ \dots & \lambda_2 D_2^* + \phi_2(1 - P_2^* - D_2^*) - \beta_1 + \beta_1 P_1^* - \beta_2 + 2\beta_2 P_2^* - \mu_2 & \lambda_1 P_1^* + \lambda_2 P_2^* & \dots \\ \dots & \beta_1 - \beta_1 P_1^* + \beta_2 - 2\beta_2 P_2^* - \lambda_2 D_2^* & -\lambda_1 P_1^* - \lambda_2 P_2^* - \delta_2(1 - P_2^* - D_2^*) - \mu_2 & \dots \end{bmatrix}$$

Then we evaluate the Jacobian at $X_1^* = (1, 0, 1, 0)$, the case when all teachers and students are positive once the system has reached equilibrium:

$$J_{1,0,1,0} = \begin{bmatrix} \beta_1 - \mu_1 & \lambda_1 + \lambda_2 & \beta_2 & 0 \\ -\beta_1 & -\lambda_1 - \lambda_2 - \mu_1 & -\beta_2 & 0 \\ \beta_1 & 0 & \beta_2 - \mu_2 & \lambda_1 + \lambda_2 \\ -\beta_1 & 0 & -\beta_2 & -\lambda_1 - \lambda_2 - \mu_2 \end{bmatrix}$$

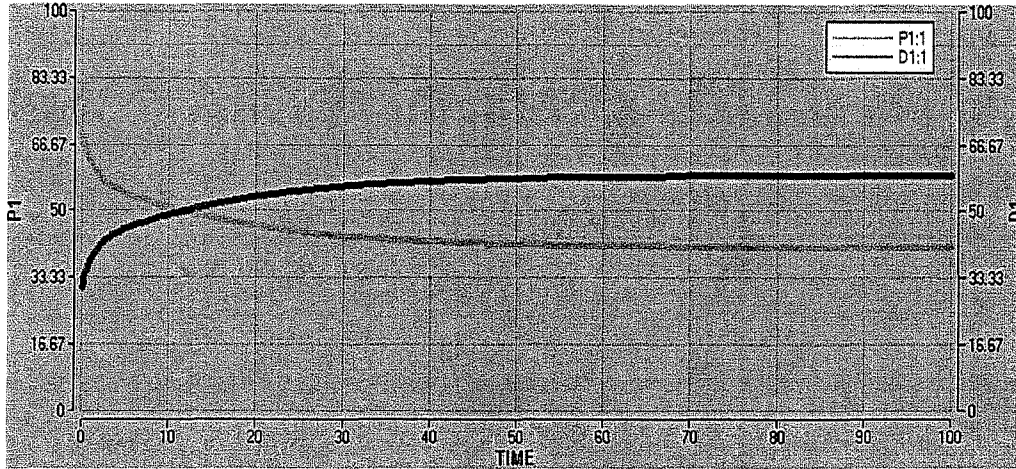


Figure 9: $r=19$, $\mathfrak{R}_0=0.93$

The trace is negative when

$$\frac{\beta_1 + \beta_2}{2\lambda_1 + 2\lambda_2 + 2\mu_1 + 2\mu_2} < 1$$

and the determinant is positive when

$$\frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} < 1.$$

But

$$\frac{\beta_1 + \beta_2}{2\lambda_1 + 2\lambda_2 + 2\mu_1 + 2\mu_2} < 1 \text{ whenever } \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} < 1.$$

Thus, the discouragement free equilibrium point is locally asymptotically stable when the condition for the determinant to be positive is satisfied.

4.1 The Basic Discouragement Number for the Three Class Model

As with the two class model, to determine the basic discouragement number, \mathfrak{R}_0 , for the discouragement free equilibrium in the three class model, we use the method outlined by van den Driessche and Watmouth (Driessche 2002). The details are found in appendix 2. The result is that for the three class model,

$$\mathfrak{R}_0 = \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2}.$$

This is the same condition we found for the stability of the discouragement free equilibrium point for both the three class model and two class models, and clearly, it is the same value we defined as the discouragement free number for the two class model. This tells us that the addition of the reluctant class to the two class model does not affect the dynamics of the two class model very much.

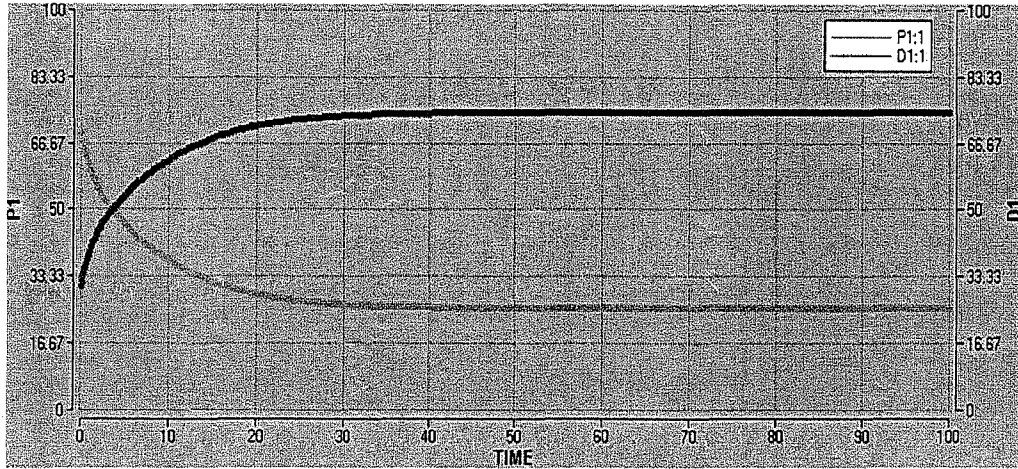


Figure 10: $r=21, \mathfrak{R}_0=1.03$

4.2 Numerical Simulations for the Three Class Model

| Teacher Parameters | Estimation | Student Parameters | Estimation |
|--------------------|---------------------------------|--------------------|---------------------------------|
| P_1 | 60 | P_2 | 350 |
| D_1 | 30 | D_2 | 350 |
| R_1 | 10 | R_2 | 300 |
| β_1 | $\left(\frac{r-1}{1+3r}\right)$ | β_2 | $\left(\frac{r-1}{1+2r}\right)$ |
| λ_1 | $e^{-0.6r}$ | λ_2 | $e^{-0.5r}$ |
| μ_1 | 0.121 | μ_2 | 0.004 |
| δ_1 | 0.3 | δ_2 | 0.2 |
| ϕ_1 | 0.01 | ϕ_2 | 0.06 |
| q_1 | 0.7 | q_2 | 0.5 |

Table 2: Initial Conditions

Using specific initial conditions obtained from our data (see Table 2) and by looking at a student teacher ratio of 6, we acquire the following graph, figure 11:

At this value for r , the number of positive students is much greater than that of discouraged students. But by changing the student teacher ratio by one (see 12), the number of reluctant students overcome the number of positive students.

In both of these cases, the value for \mathfrak{R}_0 is less than one. Since positive students and negative (R_2 and D_2) students exist simultaneously, this indicates a backward bifurcation.

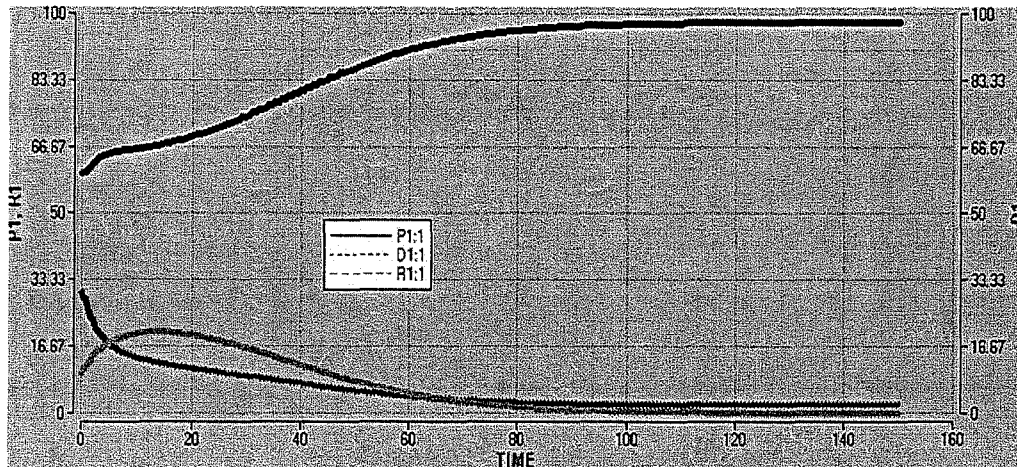


Figure 11: $r=6$, $\mathcal{R}_0=.45$

5 Results and Discussion

We were able to derive the basic discouragement number, \mathcal{R}_0 for the two class model. It turns out that the two class and three class models share the same condition for the discouragement free equilibrium points in the respective models to be locally asymptotically stable. The discouragement free equilibrium points are locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable otherwise.

Using the center manifold theory, we proved the existence of a backward bifurcation in the two class model. This means when $\mathcal{R}_0 < 1$ and $C(r) < 1$, the condition for the existence of the backward bifurcation is satisfied. So we have the disease free equilibrium being stable, an unstable saddle point endemic equilibrium, and a stable endemic equilibrium. We confirmed the existence of two stable equilibrium points by looking at the basin of attraction when the conditions are satisfied for the existence of a backward bifurcation. We do this in place of being able to analytically identify the equilibria and verify their stability.

For the backward bifurcation there is the condition

$$1 > \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu_2} = C(r).$$

When $\mathcal{R}_0 < 1$, on average a discouraged individual discourages less than one positive individual. Since, there exists a backward bifurcation for this two class model even when $\mathcal{R}_0 < 1$ there will always have a quantity of discouraged individuals in this setting. If $\mathcal{R}_0 > 1$ each negative individual converts on average more than one positive individual to a negative individual. The negative behavior will overcome the population.

We also computed sensitivity analysis for the basic discouragement number. It shows that

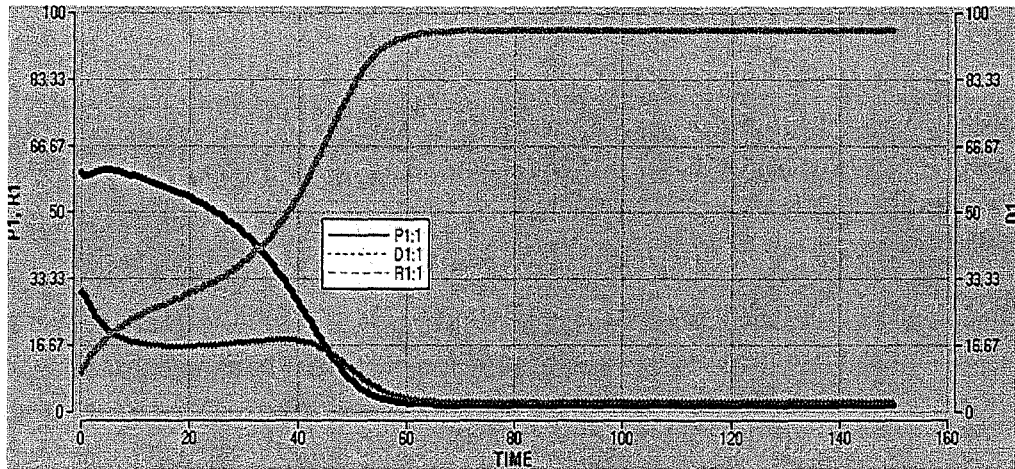


Figure 12: $r=7$, $\mathcal{R}_0=.49$

the encouragement and discouragement rates of students have the greatest impact on the basic discouragement number for the population.

Our simulations confirm that at the initial conditions essential for a backward bifurcation, a stable epidemic equilibrium point is present. At this point, when the student-teacher ratio is less than 19, the proportions of positive students and teachers remain greater than those of discouraged students and teachers. If the student teacher ratio is greater than 19, the proportions of discouraged students and teachers remain greater than those of positive students and teachers. In the educational setting, this implies that classes should have a student-teacher ratio be less than 19 for positive students and teachers to have the most influential attitudes in the classroom.

Thus far, we see many similar results from our two class model in our three class model. Moreover, our simulations for the three class model suggests there may exist a backward bifurcation.

From the results, we have gathered thus far in this model we have been able to begin to see how student-teacher ratio does affect the performance of a teacher and a student. We are continuing to study the dynamics in a classroom through the three class model. We have seen how the student-teacher ratio does affect the attitudes of the teachers and the students. Through simulations, we have found that there may be a point where the discouraged attitudes prevail over the positive attitudes, which is also reflected by \mathcal{R}_0 .

When we add the reluctance class into the model, the contribution of the students to the discouragement effect in the system is greater.

6 Conclusion

There have been many statistical studies on student-teacher ratio and how effective lowering student-teacher ratios are in the schools that choose reform through class size reduction. While it remains a controversial issue, it is important to explore what the potential positive impact lowering student-teacher ratio below a critical value could have on the success of the educational system.

This study integrates students' and teachers' interactions as an important factor for the propagation of discouragement. The simulations demonstrated that student-teacher ratio and interactions have a strong effect on the attitudes of the high school classroom population. Attitudes and student-teacher ratio affects students' and teachers' performance, since the factors that affect the student or teacher populations are not unique to either of them. In our study, we show that a class with a student-teacher ratio of under 19 is the most beneficial setting for a classroom. Even though an amount of discouraged and reluctant individuals will always exist in the populations, educators can focus on finding ways to minimize the student-teacher ratio and look for methods to help encourage themselves and students to help the performance of the entire school population.

7 Future Work

We plan to further analyze the three class model by working on the possible endemic equilibria and their stability. We also plan to survey more high schools in order to find more data to represent situations in different school districts. We would like to take into account teacher salary, school funding, and after school programs. In return, with this new information we want to link this model more strongly to an educational setting.

8 Acknowledgments

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A Basic Discouragement Number for Three Class Model

We begin with the discouraged equations because we are trying to find the \mathfrak{R}_0 that does not include positive or reluctant classes.

$$\begin{aligned}\dot{D}_1 &= \mu_1(1 - q_1) + \beta_1(r)P_1(1 - P_1) + \beta_2(r)P_1(1 - P_2) \\ &\quad - \lambda_1(r)P_1D_1 - \lambda_2(r)P_2D_1 - \delta_1R_1D_1 - \delta_2R_2D_1 - \mu_1D_1 \\ \dot{D}_2 &= \mu_2(1 - q_2) + \beta_1(r)P_2(1 - P_1) + \beta_2(r)P_2(1 - P_2) \\ &\quad - \lambda_1(r)P_1D_2 - \lambda_2(r)P_2D_2 - \delta_1R_1D_2 - \delta_2R_2D_2 - \mu_2D_2.\end{aligned}$$

We form the matrix \tilde{F} with all the terms in the equations \dot{D}_1 and \dot{D}_2 representing the population going into the discouraged classes. Thus, the first row of the matrix \tilde{F} is the terms of \dot{D}_1 and the second row is the terms of \dot{D}_2 :

$$\tilde{F} = \begin{pmatrix} \mu_1 - \mu_1q_1 + \beta_1P_1D_1 + \beta_1P_1R_1 + \beta_2P_1D_2 + \beta_2P_1R_2 \\ \mu_2 - \mu_2q_2 + \beta_1P_2D_1 + \beta_1P_2R_1 + \beta_2P_2D_2 + \beta_2P_2R_2 \end{pmatrix}.$$

Then we form the matrix \tilde{V} with the negative of the terms from \dot{D}_1 in the first row and the negative of the terms from \dot{D}_2 in the second row. These terms represent the population leaving the discouraged classes:

$$\tilde{V} = \begin{pmatrix} \lambda_1P_1D_1 + \lambda_2P_2D_1 + \delta_1R_1D_1 + \delta_2R_2D_1 + \mu_1D_1 \\ \lambda_1P_1D_2 + \lambda_2P_2D_2 + \delta_1R_1D_2 + \delta_2R_2D_2 + \mu_2D_2 \end{pmatrix}.$$

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Now we take the partial derivatives of \tilde{F} and of \tilde{V} :
 The partial derivatives for \tilde{F} are:

$$\frac{\partial D_1}{\partial D_1} = \beta_1 P_1$$

$$\frac{\partial D_1}{\partial D_2} = \beta_2 P_1$$

$$\frac{\partial D_2}{\partial D_2} = \beta_2 P_1$$

$$\frac{\partial D_2}{\partial D_1} = \beta_1 P_2.$$

The partial derivatives for \tilde{V} are:

$$\frac{\partial D_1}{\partial D_1} = \lambda_1 P_1 + \lambda_2 P_2 + \delta_1 R_1 + \delta_2 R_2 + \mu_1$$

$$\frac{\partial D_1}{\partial D_2} = 0$$

$$\frac{\partial D_2}{\partial D_2} = \lambda_1 P_1 + \lambda_2 P_2 + \delta_1 R_1 + \delta_2 R_2 + \mu_2$$

$$\frac{\partial D_2}{\partial D_1} = 0.$$

The next step is to let $R_i = 0$ and $D_i = 0$ since R_i represent the reluctant classes and D_i represent the discouraged classes. This way we remove the negative classes. The partial derivatives of \tilde{F} stay the same, while the partial derivatives of \tilde{V} have some changes:

$$\frac{\partial D_1}{\partial D_1} = \lambda_1 P_1 + \lambda_2 P_2 + \mu_1$$

$$\frac{\partial D_1}{\partial D_2} = 0$$

$$\frac{\partial D_2}{\partial D_2} = \lambda_1 P_1 + \lambda_2 P_2 + \mu_2$$

$$\frac{\partial D_2}{\partial D_1} = 0$$

Now, we have the matrices F and V formed from the partials of all the terms in the matrices \tilde{F} and \tilde{V} without any terms of R_i and D_i :

$$F = \begin{pmatrix} \beta_1 P_1 & \beta_2 P_1 \\ \beta_1 P_2 & \beta_2 P_2 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \lambda_1 P_1 + \lambda_2 P_2 + \mu_1 & 0 \\ 0 & \lambda_1 P_1 + \lambda_2 P_2 + \mu_2 \end{pmatrix}.$$

Substituting $(P_1, D_1, R_1, P_2, D_2, R_2) = (1, 0, 0, 1, 0, 0)$, for F we get

$$F = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

and for V we get

$$V = \begin{pmatrix} \lambda_1 + \lambda_2 + \mu_1 & 0 \\ 0 & \lambda_1 + \lambda_2 + \mu_2 \end{pmatrix}.$$

The next step in this process is to find V^{-1} ,

$$V^{-1} = \begin{pmatrix} \frac{1}{\lambda_1 + \lambda_2 + \mu_1} & 0 \\ 0 & \frac{1}{\lambda_1 + \lambda_2 + \mu_2} \end{pmatrix}.$$

Now we multiply F and V^{-1} :

$$FV^{-1} = \begin{pmatrix} \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \\ \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \end{pmatrix}.$$

This leaves us with the Jacobian matrix to use in order to determine the system's dominant eigenvalue:

$$(J - \Lambda) = \begin{pmatrix} \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} - \Lambda & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \\ \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} & \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} - \Lambda \end{pmatrix}.$$

From this matrix we compute the characteristic equation and the eigenvalues.

$$\begin{aligned} \det(J - \Lambda) &= \left(\frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} - \Lambda \right) \left(\frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} - \Lambda \right) - \left(\frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} \right) \left(\frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \right) \\ \Rightarrow 0 &= \left(\frac{\beta_1 - \Lambda(\lambda_1 + \lambda_2 + \mu_1)}{\lambda_1 + \lambda_2 + \mu_1} \right) \left(\frac{\beta_2 - \Lambda(\lambda_1 + \lambda_2 + \mu_2)}{\lambda_1 + \lambda_2 + \mu_2} \right) - \left(\frac{\beta_1 \beta_2}{(\lambda_1 + \lambda_2 + \mu_1)(\lambda_1 + \lambda_2 + \mu_2)} \right) \\ 0 &= \Lambda^2 - \frac{\beta_1 \Lambda}{\lambda_1 + \lambda_2 + \mu_1} - \frac{\beta_2 \Lambda}{\lambda_1 + \lambda_2 + \mu_2} \\ 0 &= \Lambda \left(\Lambda - \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} - \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} \right) \\ \Lambda &= 0 \text{ or } \Lambda = \frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2}. \end{aligned}$$

Then,

$$\Re_0 = \max(\Lambda).$$

In this case the maximum value of the eigenvalues is

$$\frac{\beta_1}{\lambda_1 + \lambda_2 + \mu_1} + \frac{\beta_2}{\lambda_1 + \lambda_2 + \mu_2} = \Re_0.$$

B Codes

In order to work on the simulations we created programs in MATLAB, Berkeley Madonna, and DYNAMICS that were composed basically of differential equations, the data found, and the plotting of the graphs.

B.1 Berkeley Madonna

This is the code for Berkeley Madonna simulations. METHOD RK4

```
STARTTIME = 0
STOPTIME=100
DT = 0.02
```

```
lambda1=EXP(-.06*r)
lambda2=EXP(-.05*r)
beta1=(r-1)/(1+3*r)
beta2=(r-1)/(1+2*r)
```

```
d/dt(P1)=mu1*q1*N1+lambda1*D1*P1/N1+lambda2*D1*P2/N2-beta1*P1*D1/N1
-beta2*P1*D2/N2-mu1*P1
d/dt(D1)=mu1*N1-mu1*q1*N1+beta1*P1*D1/N1+beta2*P1*D2/N2
-lambda1*D1*P1/N1-lambda2*D1*P2/N2-mu1*D1
d/dt(P2)=mu2*q2*N2+lambda1*D2*P1/N1+lambda2*D2*P2/N2-beta1*P2*D1/N1
-beta2*P2*D2/N2-mu2*P2
d/dt(D2)=mu2*N2-mu2*q2*N2+beta1*P2*D1/N1+beta2*P2*D2/N2-lambda1*D2*P1/N1
-lambda2*D2*P2/N2-mu2*D2
```

```
RR=beta1/(lambda1+lambda2+mu1) +beta2/(lambda1+lambda2+mu1)
C=lambda1/(lambda1+lambda2+mu1)+lambda2/(lambda1+lambda2+mu1)
```

```
init P1=69
init D1=31
init P2=500
```

init D2=500

r=10/1
N1=100
N2=1000
mu1=.121
mu2=.004
q1=.7
q2=.5

METHOD RK4

STARTTIME = 0
STOPTIME=150
DT = 0.02

lambda1=EXP(-.06*r)
lambda2=EXP(-.05*r)
beta1=(r-1)/(1+3*r)
beta2=(r-1)/(1+2*r)

$d/dt(P1) = N1 * mu1 * q1 + lambda1 * D1 * (P1/N1) + lambda2 * D1 * (P2/N2) - beta1 * P1 + beta1 * P1 * (P1/N1) - beta2 * P1 + beta2 * P1 * (P2/N2) + phi1 * R1 * (P1/N1) + phi2 * R1 * (P2/N2) - mu1 * P1$

$d/dt(D1) = beta1 * P1 - beta1 * P1 * (P1/N1) + beta2 * P1 - beta2 * P1 * (P2/N2) - lambda1 * D1 * (P1/N1) - lambda2 * D1 * (P2/N2) - mu1 * D1 - delta1 * D1 * (R1/N1) - delta2 * D1 * (R2/N2) + mu1 * N1 - mu1 * q1 * N1$

$d/dt(R1) = delta1 * D1 * (R1/N1) + delta2 * D1 * (R2/N2) - phi1 * R1 * (P1/N1) - phi2 * R1 * (P2/N2) - mu1 * R1$

$d/dt(P2) = N2 * mu2 * q2 + lambda2 * (P2/N2) * D2 + lambda1 * (P1/N1) * D2 - beta1 * P2 + beta1 * P2 * (P1/N1) - beta2 * P2 + beta2 * P2 * (P2/N2) + phi1 * R2 * (P1/N1) + phi2 * R2 * (P2/N2) - mu2 * P2$

$d/dt(D2) = beta1 * P2 - beta1 * P2 * (P1/N1) + beta2 * P2 - beta2 * P2 * (P2/N2) - lambda1 * (P1/N1) * D2 - lambda2 * D2 * (P2/N2) - delta1 * D2 * (R1/N1) - delta2 * D2 * (R2/N2) - mu2 * D2 + mu2 * N2 - mu2 * q2 * N2$

$d/dt(R2) = delta1 * D2 * (R1/N1) + delta2 * D2 * (R2/N2) - phi1 * R2 * (P1/N1) - phi2 * R2 * (P2/N2) - mu2 * R2$

$RR = beta1 / (lambda1 + lambda2 + mu1) + beta2 / (lambda1 + lambda2 + mu1)$

r=20

init P1=60
init D1=30
init R1=10
init P2=350
init D2=350
init R2=300

N1=100
N2=1000
mu1=0.004
mu2=0.121
q1=0.7
q2=0.5
delta1=.3
delta2=.2
phi1=.01
phi2=.06

B.2 Tables of Data and Parameters

B.3 DYNAMICS

For the simulations made in DYNAMICS we studied the case of discourage free equilibria ($q_1 = q_2 = 1$). The following picture shows the existence of two attractors, which means the existence of a backward bifurcation under the given condition.

| State | District | Number of Students | Instructional Expenditure per Student(\$) | Student and Staff Support per Student(\$) | Student/Teacher Ratio |
|-------|---------------------------|--------------------|---|---|-----------------------|
| TX | Zephyr Isd | 160 | 4,557 | 228 | 11.8 |
| TX | Masonic Home Isd | 163 | 10,410 | 1,124 | 7.4 |
| NY | Falconer Csd | 1,435 | 4,907 | 792 | 13.9 |
| NY | Bronxville Ufsd | 1,466 | 9,357 | 1,495 | 11.5 |
| MN | Pelican Rapids | 1,260 | 3,869 | 268 | 14.8 |
| MN | Cass Lake-bena Schools | 1,190 | 5,348 | 665 | 11.9 |
| NM | Aztec Municipal Schools | 3,379 | 3,038 | 661 | 16.2 |
| NM | Bernalillo Public Schools | 3,451 | 4,036 | 1,288 | 12.1 |
| CA | Riverbank Unified | 3,221 | 3,454 | 0 | 20.6 |
| CA | Albany City Unified | 3,020 | 4,718 | 0 | 19.1 |

Table 3: Comparison of Similar Districts Within Five States



Figure 13: $q_1 = q_2 = 1$