

Understanding Immigration and Policy Change from a Mathematical Perspective

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Abstract

Every year nearly 300,000 Mexican people cross the border into the United States. This paper is a discussion of the effect of changes in United States policy on Mexican immigration rates. Using three separate compartmental models, we look at Mexican immigration as a whole and conclude immigration is inevitable; a model of the effect of quotas on illegal Mexican immigration; and the effect of the Patriot Act.

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Introduction

The United States is a country founded by immigrants. From its inception in the late 1700s to the 1880s, immigrants came to the United States from varied locales with little legal restriction. As the number of people wishing to immigrate to the United States increased, the government was forced to take control of immigration and place restrictions on those allowed residency. The Chinese Exclusionary Act was passed in 1882 and marked the first in a series of exclusionary acts which limited the number and nationality of immigrants permitted residence. These acts as well as subsequent immigration laws have had a profound impact on the flow of individuals coming into the United States, both legally and illegally.

Socioeconomic hardships are common in Mexican populations and encourage a large number of Mexican people to attempt to emigrate to the United States in search of employment. Throughout the United States (save New England), people from Mexico comprise the most prevalent immigrant group. In some areas, the immigrant population from Mexico accounts for more than a quarter of the total population. Current estimates of undocumented immigrant population consistently estimate over half of all illegal immigrants are of Mexican descent. Given the prevalence of Mexican immigration, it is crucial to understand how they are affected by the United States immigration policy.

We have obtained data on immigration from various sources. For the number of legal immigrants from Mexico, we used the United States census. Estimates of illegal immigrants were taken from Cornelius, whose estimates were based on the number of apprehensions; Bean et al, whose estimates were based on estimates of the illegal work force; and Massey et al., whose estimates were based on unclaimed deaths of suspicious nature (i.e. dehydration in the desert, drowning in tributaries, etc.). Using this data, we will attempt to glean the effect of changes in United States immigration policy on the flow of both legal and illegal Mexican immigrants into the United States.

Models of Immigration

To investigate how Mexican immigration is affected by policy change, we first explore a simple compartmental model of Mexican immigration considering three classes of people—individuals in Mexico, individuals in Mexico wishing to emigrate, and Mexican immigrants in the United States. In a second model, we divide the class of Mexican immigrants in the United States into two further classes—those who came to the United States legally with a visa, and those who reside in the country illegally. We use this model to explore how the quota on the number of visa allotted to Mexican immigrants effects the ratio at which legal to illegal immigrants enter the United States. In a third and final model we consider an additional class of people in Mexico—illegal Mexican immigrants that have returned to Mexico either voluntarily or via deportation. Using this model, we explore how the Patriot Act effected the ratio of legal to illegal Mexican immigrants in the United States.

A Model of Immigration

In this model, we focus on three classes: individuals in Mexico susceptible to persuasion to try and emigrate to the United States (S), individuals in Mexico who have been influenced by the idea of coming to the United States (M_1), and individuals who have already emigrated to the United States (M_2). Individuals in the M_1 class will emigrate to the United States at some rate δ_1 . Similarly those in the M_2 class will return to the United States at another rate δ_2 . People in the susceptible class are introduced to the idea of coming to the United States by individuals who have already thought about coming to the United States (M_1), or individuals who have already come to the United States (M_2). Individuals who have already been to the United States are going to be more influential in convincing people in the S class. This increase in influence is measured by ρ (which we assume to be greater than 1). Putting this together, we have the compartmental model described in Figure 1.

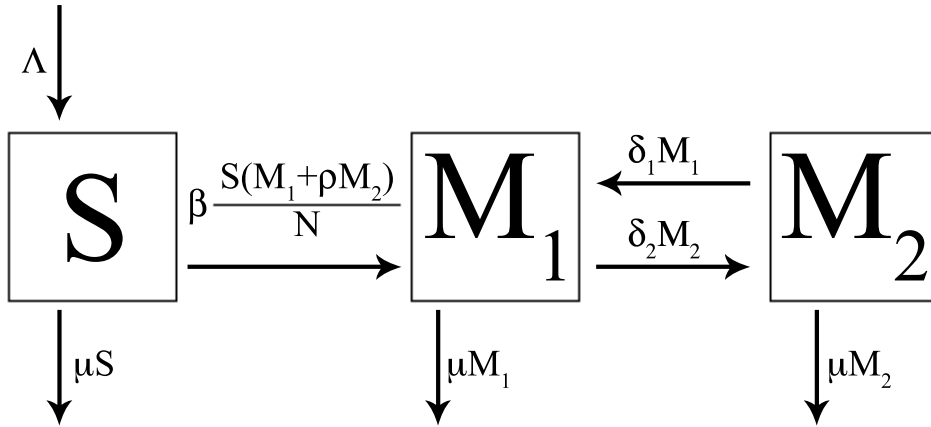


Figure 1: Simple Model of Mexican Immigration

| Parameter and classes | Distinction |
|-----------------------|--|
| S | Individuals in Mexico that have not considered emigrating to the United States |
| M_1 | Individuals in Mexico that intend to immigrate to the United States |
| M_2 | Mexican immigrants in the United States |
| Λ | Birth rate |
| μ | Death rate |
| δ_1 | Rate at which individuals emigrate to the United States |
| δ_2 | Rate at which immigrants in the United States return to Mexico |
| β | Force of influence |
| ρ | Proportional influence of M_2 class |

Table 1: Parameters for Simple Model

The rate at which Mexican people want to come to the US is δ_1 ; therefore, $\delta_1 M_1$ represents the number of Mexican people who actually come to the United States per unit time, or enter M_2 class. Similarly, δ_2 is the rate at which immigrants in the US want to return to Mexico; therefore, $\delta_2 M_2$ is the number of Mexican immigrants that return to Mexico, or reenter M_1 class per unit time. The rate at which people are born into the system is Λ and natural

death is represented by μ .

Movement from the S class into the M_1 class occurs at some rate $\beta S \frac{(M_1 + \rho M_2)}{N}$. β is equal to the probability of being convinced to emigrate multiplied by the number of contacts per time. $\frac{(M_1 + \rho M_2)}{N}$ is the proportion of the population capable of convincing susceptible individuals to emigrate. $\beta \frac{(M_1 + \rho M_2)}{N}$ has units of per time. Multiplying this by S , which is the total susceptible population, will yield people per time, which are the units we want. From this model we obtain the following system of equations:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta S \frac{M_1 + \rho M_2}{N} - \mu S \\ \frac{dM_1}{dt} &= \beta S \frac{M_1 + \rho M_2}{N} - \mu M_1 - \delta_1 M_1 + \delta_2 M_2 \\ \frac{dM_2}{dt} &= \delta_1 M_1 - \delta_2 M_2 - \mu M_2\end{aligned}\tag{1}$$

For our analysis, we will assume that the population is constant, thus $\Lambda = \mu N$. We can find two equilibrium—the immigrant free equilibrium (IFE) $(S^*, M_1^*, M_2^*) = (N, 0, 0)$, and an equilibrium with immigrants (IE):

$$\begin{aligned}S^* &= \frac{\mu N(\delta_1 + \delta_2 + \mu)}{\beta(\delta_2 + \mu + \rho\delta_1)} \\ M_1^* &= -\frac{N(\delta_1\mu + \mu\delta_2 + \mu^2 - \beta\delta_2 - \beta\mu - \beta\rho\delta_1)(\delta_2 + \mu)}{\beta(\delta_2\delta_1 + \delta_2^2 + 2\mu\delta_2 + \delta_1\mu + \mu^2 + \rho\delta_1^2 + \rho\delta_1\delta_2 + \rho\delta_1\mu)} \\ M_2^* &= -\frac{\delta_1 N(\delta_1\mu + \mu\delta_2 + \mu^2 - \beta\delta_2 - \beta\mu - \beta\rho\delta_1)}{\beta(\delta_2\delta_1 + \delta_2^2 + 2\mu\delta_2 + \delta_2\mu + \mu^2 + \rho\delta_1^2 + \rho\delta_1\delta_2 + \rho\delta_1\mu)}\end{aligned}$$

In order for the equilibria to be stable, we linearize the system about the equilibria and ensure that all the eigenvalues of this Jacobian are negative. For this system, our Jacobian is:

$$J|_{IFE} = \begin{bmatrix} -\frac{\beta(M_1 + \rho M_2)}{N} - \mu & -\frac{\beta S}{N} & -\frac{\beta S \rho}{N} \\ \frac{\beta(M_1 + \rho M_2)}{N} & \frac{\beta S}{N} - \delta_1 - \mu & \frac{\beta S \rho}{N} + \delta_2 \\ 0 & \delta_1 & -\delta_2 - \mu \end{bmatrix}$$

Evaluated at the immigration-free equilibrium:

$$\begin{bmatrix} -\mu & -\beta & -\beta\rho \\ 0 & \beta - \delta_1 - \mu & \beta\rho + \delta_2 \\ 0 & \delta_1 & -\delta_2 - \mu \end{bmatrix}$$

We now want to look at the eigenvalues of this matrix. If they are all negative, our equilibrium will be stable. Expanding $\det(J - \lambda I)$ about the first column, we obtain

$$(-\mu - \lambda) \begin{vmatrix} \beta - \delta_1 - \mu - \lambda & \beta\rho + \delta_2 \\ \delta_1 & -\delta_2 - \mu - \lambda \end{vmatrix}$$

Thus we have one eigenvalue equal to $-\mu$, and we have reduced it to a two dimensional case. We can use Routh-Hurwitz criterion to determine the sign of the eigenvalues. Our determinate, then, must be positive and our trace negative, or $\beta - \delta_1 - \mu - \delta_2 - \mu < 0$ and $(\beta - \delta_1 - \mu)(-\delta_2 - \mu) - (\delta_1)(\beta\rho + \delta_2) > 0$. This means that the disease free equilibrium is stable when $\beta < \delta_1 + \delta_2 - 2\mu$ and $(\beta - \delta_1 - \mu)(-\delta_2 - \mu) > (\delta_1)(\beta\rho + \delta_2)$. Thus, when $\frac{(\delta_1)(\beta\rho + \delta_2)}{(\beta - \delta_1 - \mu)(-\delta_2 - \mu)} < 1$ the disease free equilibrium is stable.

Using the criteria that guarantees the stability of the IFE as our R_0 and rearranging the terms, this gives us $R_0 = \frac{\beta}{\mu + \delta_1} + \frac{\delta_1}{\mu + \delta_1} \frac{\delta_2}{\mu + \delta_2} \frac{\delta_1}{\mu + \delta_1} \frac{\delta_2}{\mu + \delta_2} + \frac{\delta_1}{\mu + \delta_1} \frac{\beta\rho}{\mu + \delta_2}$. R_0 is a measure of the average number of individuals that are convinced to go to the United States by one individual during his/her lifespan. $\frac{\beta}{\mu + \delta_1}$ is the proportion of individuals convinced to emigrate by the M_1 class. Next we have $\frac{\delta_1}{\mu + \delta_1} \frac{\delta_2}{\mu + \delta_2}$, which is the probability the individuals survives moving from the M_1 class to the M_2 class and back to the M_1 class again. The final term consists of $\frac{\beta\rho}{\mu + \delta_2}$ which is the proportion persuaded by the M_2 class and $\frac{\delta_1}{\mu + \delta_1}$ which is the proportion of the M_1 class that survives getting to the United States. We take the product of these terms because an individual must first get to the M_2 class before they can begin to influence people at the rate an individual in the M_2 class does.

Theorem 0.1. *If $\beta > \mu$, the immigrant free equilibrium is unstable.*

Proof. To look at the stability of the IFE, we look at R_0 as a function of δ_1 and δ_2 . Since $R_0 < 1$ gives stability of the IFE, we are particularly interested in the plane where $R_0 = 1$.

The intersection of $R_0(\delta_1, \delta_2)$ and $R_0 = 1$ occurs in a line, specifically: $\delta_2 = \frac{\beta\rho - \mu}{\mu - \beta} \delta_1 - \mu$. This line has a δ_2 intercept of $-\mu$ and a δ_1 intercept of $\frac{\mu(\beta - \mu)}{\mu - \beta\rho}$. This gives us two cases, the first where $\frac{\mu(\beta - \mu)}{\mu - \beta\rho}$ is positive and a second where it is negative.

In the first case, we either need to have the numerator and the denominator both negative or both positive. Setting them both positive, we get $\mu(\beta - \mu) > 0$ and $\mu - \beta\rho > 0$. The first inequality gives us $\beta > \mu$ and the second gives us $\mu > \beta\rho$. Since we assume that $\rho > 1$, these cannot both be true. We get a similar situation when both are negative. This means that it cannot be positive and thus must be negative. We can see the line in Figure 2

On one side of this line, $R_0 > 1$ and on the other $R_0 < 1$. Using the point $(0, 0)$ for ease of calculations, we get $R_0 = \frac{\beta}{\mu}$ at $(0, 0)$. From assuming that $\beta > \mu$ (which is sociologically sensible) we get $\frac{\beta}{\mu} > 1$ or $R_0 > 1$. This means that to the right side of the line $R_0 = 1$, $R_0 > 1$. Since we are only concerned with the case where δ_1 and δ_2 are both positive, we have that R_0 is always greater than 1, as we can see more clearly in the 3d picture in Figure 3

Since $R_0 > 1$ for all the values of parameters we are considering, the IFE is always unstable. \square

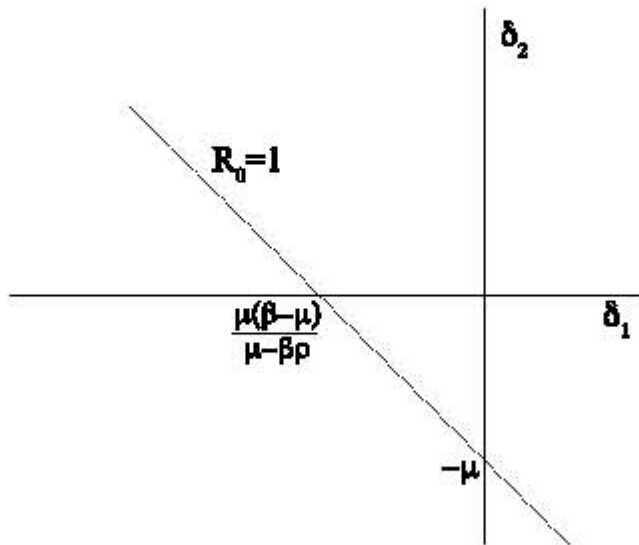


Figure 2: $\delta_2 = f(\delta_1)$

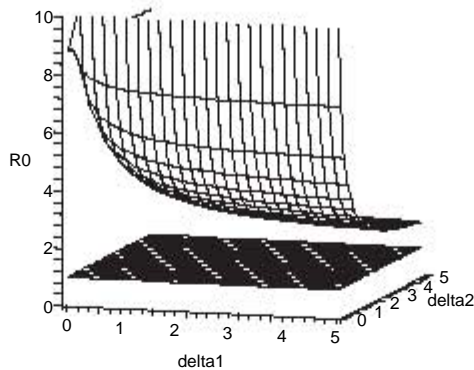


Figure 3: Intersection of $R_0(\delta_1, \delta_2)$ and $R_0 = 1$

Quota Model

In this model we have the same classes in Mexico as in the previous model. The immigrants in the United States have been divided into two classes: V are the immigrants that have visas, I are the immigrants that are in the United States illegally. Some proportion δ of individuals who want to emigrate to the United States will actually emigrate. This means that over the course of a year δL individuals will enter the United States. The United States has a quota system when allowing immigrants into the country—only a set number of immigrants φ are allowed visas in a given year. If the number of individuals entering the United States is less than this quota, then all the individuals can enter the United States

with visas. If the number that will enter the United States is greater than the quota, then φ individuals will enter the United States with visas and the remaining individuals will enter the United States illegally. This gives us the compartmental model described in Figure 4.

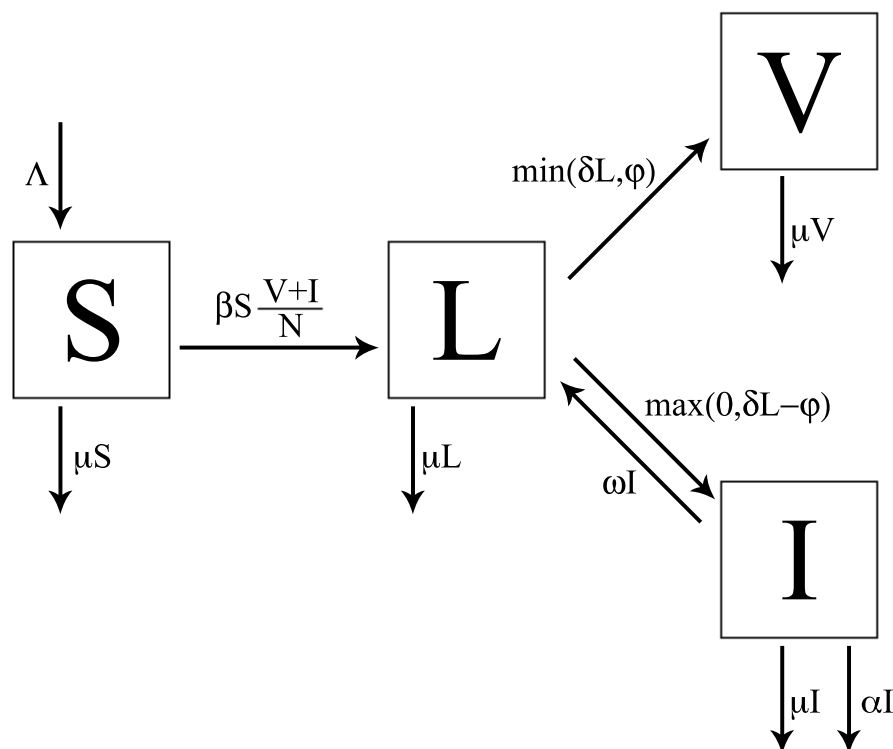


Figure 4: Quota Model

| Parameters and classes | Distinction |
|------------------------|--|
| S | Individuals in Mexico that have not considered emigrating to the United States |
| L | Individuals in Mexico that intend to emigrate to the United States |
| V | Mexican Immigrants in the United States with Visas |
| I | Mexican Immigrants in the United States who are here illegally |
| μ | Death rate |
| Λ | Birth rate |
| β | Force of influence |
| δ | Rate at which individuals emigrate to the United States |
| φ | Quota of Mexican Immigrants allowed in the US |
| ω | Rate at which illegal immigrants return to Mexico |

Table 2: Quota Model Parameters

This model has a switch at $\delta L = \varphi$, or where the number of individuals wishing to enter the country is equal to the number allowed by the quota. We again assume that individuals enter the population at a rate Λ and that death occurs at a rate μ . Movement from the S class into the L class occurs in a similar manner as the Simple Model assuming that only individuals in the United States will be able to convince people in Mexico to want to

emigrate, and that all current immigrants have the same influence on those in the S class. From this model, we obtain the following system of differential equations:

$$\begin{aligned}
 \frac{dS}{dt} &= \Lambda - \beta S \frac{V+I}{N} - \mu S \\
 \frac{dL}{dt} &= \beta S \frac{V+I}{N} - \delta L + \omega I - \mu L \\
 \frac{dI}{dt} &= \max(0, \delta L - \varphi) - \mu I - \alpha I - \omega I \\
 \frac{dV}{dt} &= \min(\varphi, \delta L) - \mu V
 \end{aligned}
 \tag{2}$$

We have δL individuals that will make it into the United States in a given year, if this quantity is less than the quota φ , all the immigrants will be allowed into the country legally. Thus when $\delta L < \varphi$ the illegal class is eliminated leaving us with the model in Figure 5.

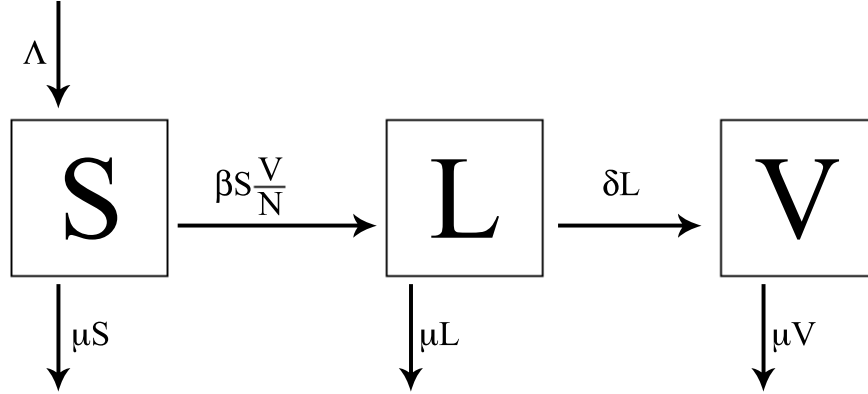


Figure 5: Quota Model Case One: $\delta L < \varphi$

In this model we have two equilibria. The immigrant free equilibria (E_1) occurs when $L^* = 0, S^* = N, V^* = 0$. The immigrant equilibrium (E_2) occurs when

$$\begin{aligned}
 S^* &= \frac{\mu N(\mu + \delta)}{\beta \delta} \\
 L^* &= \frac{\mu N(\beta \delta - \mu^2 - \delta \mu)}{\beta \delta(\mu + \delta)} \\
 V^* &= \frac{N(\beta \delta - \mu^2 - \delta \mu)}{\mu + \delta}
 \end{aligned}$$

To look at the stability of the IFE, we linearize about it, giving us:

$$J|_{IFE} = \begin{bmatrix} -\mu & 0 & -\beta \\ 0 & -\mu - \delta & \beta \\ 0 & \delta & -\mu \end{bmatrix} \quad (3)$$

The eigenvalues of this matrix are:

$$\begin{aligned} \lambda_1 &= -\mu \\ \lambda_2 &= -\mu - \frac{\delta}{2} + \frac{1}{2}\sqrt{\delta^2 + 4\beta\delta} \\ \lambda_3 &= -\mu - \frac{\delta}{2} - \frac{1}{2}\sqrt{\delta^2 + 4\beta\delta} \end{aligned}$$

λ_1 and λ_3 are both clearly negative. λ_2 is negative when $\frac{\beta\delta}{\mu(\mu+\delta)} < 1$. Thus E_1 is stable when $\frac{\beta\delta}{\mu(\mu+\delta)} < 1$

The immigrant equilibria E_2 exists when L^* , S^* , and V^* are all positive, or when $\beta\delta - \mu^2 - \delta\mu > 0$. This occurs when $\frac{\beta\delta}{\mu(\mu+\delta)} > 1$, or $R_0 > 1$. When the IE exists, it is stable.

In the second case, where $\delta L > \varphi$ we have both the legal and illegal classes, as we see in Figure 6

Since there is a constant number of individuals entering the V class, there is no immigrant free equilibrium. There is only an immigrant equilibria E_3 that exists when $1 + \frac{\varphi\beta}{\Lambda\mu} < R_0 < 1 + \frac{\varphi\beta}{\Lambda\mu} + \frac{\beta-\alpha}{\alpha}$ (see Appendix for further details).

Having analyzed each of the cases, we need to look at where the two cases meet. The two cases meet when $\delta L = \varphi$. This makes our switching point $L_{SWITCH} = \frac{\varphi}{\delta}$. When $R_0 < 1$, we have the IFE from case one, call it E_1 stable. Since in our case two model, L^* is always greater than $\frac{\varphi}{\delta}$, thus the IE of case two, call it E_3 is in the switching model when it exists. The IE of case one, call it E_2 , is in the model when it exists and when $L^* < L_{SWITCH}$. This occurs when $1 + \frac{\varphi\beta}{\Lambda\mu} > R_0$. This means when $R_0 < 1 + \frac{\varphi\beta}{\Lambda\mu}$ there are no illegal immigrants. Looking at this graphically in Figure 7.

To gain further insight into the Quota Model, we decided to run numerical simulations. Using MatLab, we ran simulations for various values of φ in Figure 8 .

As we can see, as the quota increases, the number of people in the V class increases, since more immigrants are allowed into the country. The I class, however, does not get smaller. This shows that as you increase the quota, you increase the total number of immigrants in the country. We want to look at the amount the number of immigrants changes as we make changes to the quota. In Figure 9

From here we can see that as φ gets larger, the same increase in φ will result in a smaller

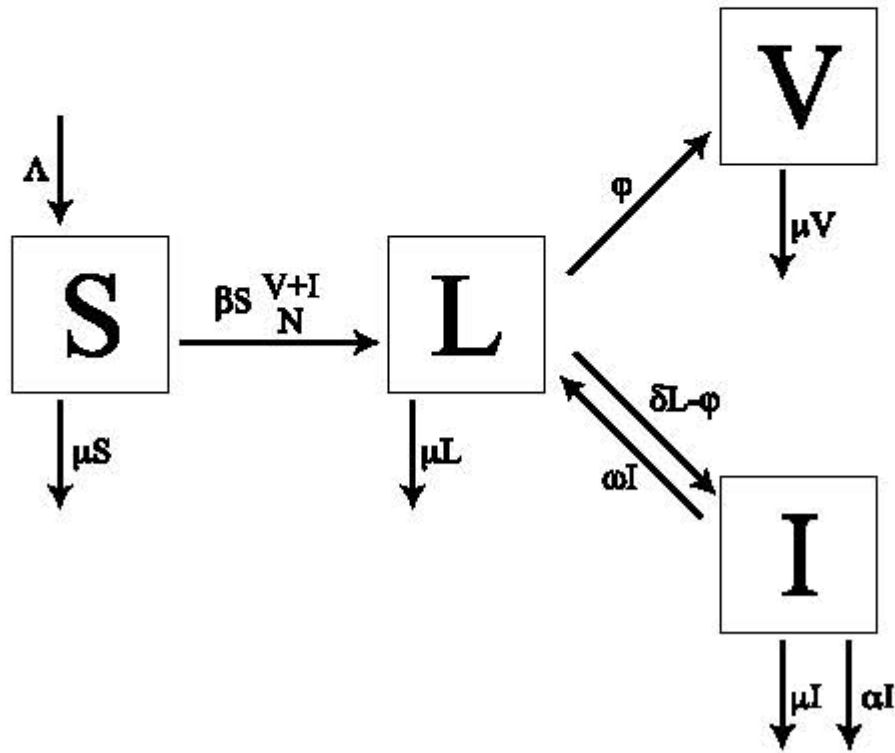


Figure 6: Quota Model Case Two: $\delta L > \varphi$

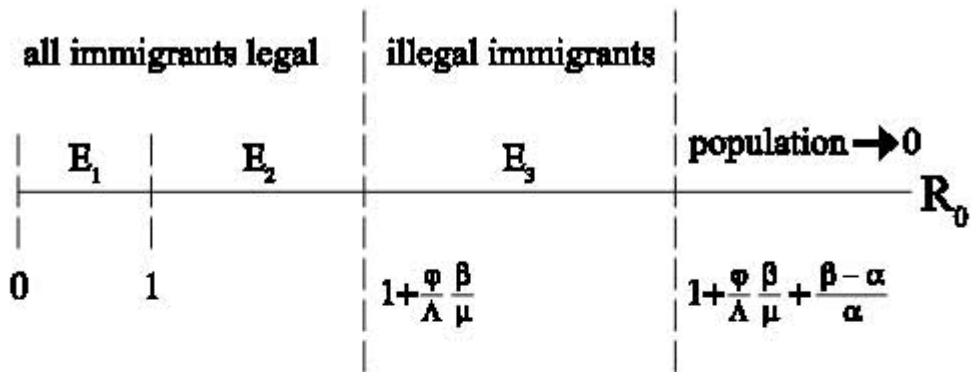


Figure 7: R_0 and Stability of Equilibria

increase in the total Mexican immigrant population.

Initially, we thought that increasing the quota would help to eliminate illegal immigrant death due to hazards of crossing the border (increasing the number allowed in legally would seem to decrease the number that have to come in illegally). Since the opposite was true—increasing the quota actually increased the number of illegal immigrants, it is also interesting to look at how changing the quota effects the number of immigrant deaths, which we can

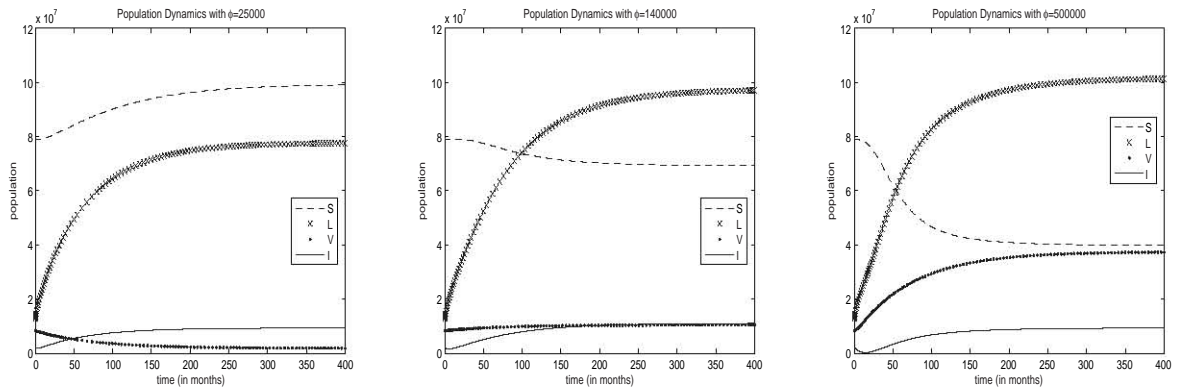


Figure 8: Numerical Simulation with varying quotas

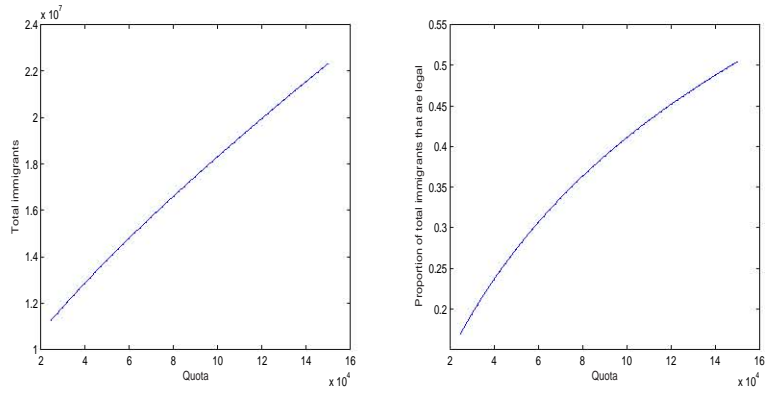


Figure 9: Quota versus total immigrants and proportion that are legal

see in Figure 10.

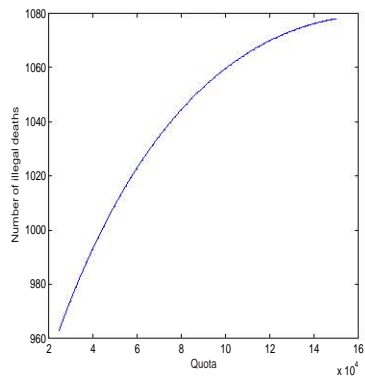


Figure 10: Quota versus number of illegal deaths

Since increasing the quota increases the size of the illegal classes, it also increases the number of illegal deaths.

9/11 Model

The classes in this model are again similar to those in the previous model. The only new class is R which is the class of illegal immigrants that have returned to Mexico. We have δL individuals that cross the border from Mexico into the United States and some proportion q do it legally, while the remaining $1 - q$ illegally. Those that enter the United States illegally leave the country voluntarily at some rate τ and a further number are removed through deportation at a rate ω . Those illegal immigrants that have returned to Mexico will return to the US again (illegally) at some rate σ . This leaves us with the model in Figure 11

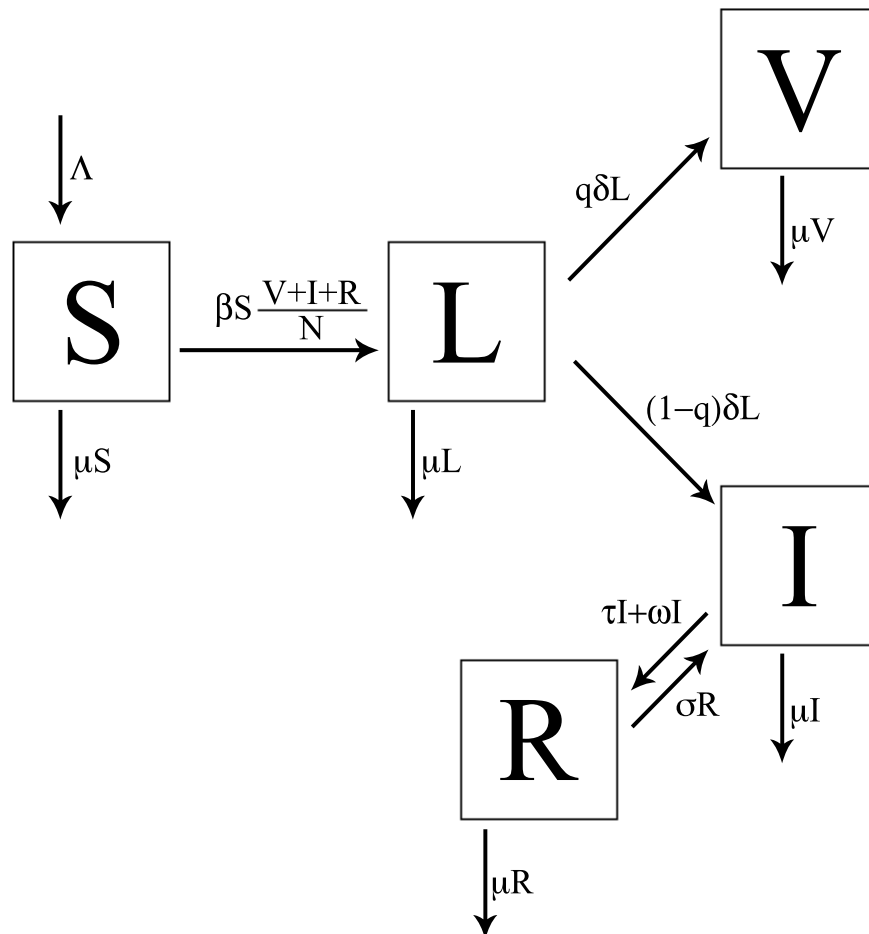


Figure 11: Model of Mexican Immigration

Individuals are born into the population at a rate Λ and leave the population at a rate μ . Susceptible individuals are brought into the L class through contact with individuals who have already been to the United States in the same manner as the previous model.

The system has two equilibria. The IFE is $L^* = 0, S^* = N, I^* = 0, R^* = 0, V^* = 0$. The

| Parameters and Classes | Distinction |
|------------------------|--|
| S | Individuals in Mexico that have not considered emigrating to the United States |
| L | Individuals in Mexico that intend to emigrate to the United States |
| V | Mexican Immigrants in the United States with Visas |
| I | Mexican Immigrants in the United States who are here illegally |
| R | Illegal Mexican immigrants that have returned to Mexico |
| Λ | Birth rate |
| μ | Natural death rate |
| δ | Rate at which individuals emigrate to the United States |
| β | Force of influence |
| q | Proportion of immigrants that get into the United States legally |
| ω | Rate at which illegal immigrants are removed due to deportation |
| τ | Rate at which illegal immigrants voluntarily move to Mexico |
| σ | Rate at which those in the R class return to the United States |

immigrant equilibria occurs when

$$\begin{aligned}
R^* &= (1-p) \frac{(\beta\delta - \delta\mu - \mu^2)N(\tau + \omega)}{\beta(\delta + \mu)(\mu + \sigma + \tau + \omega)} \\
S^* &= \frac{\mu N(\delta + \mu)}{\beta\delta} \\
L^* &= \frac{\mu N(\beta\delta - \delta\mu - \mu^2)}{\beta\delta(\delta + \mu)} \\
I^* &= \frac{N(-\beta\delta + \delta\mu + \mu^2)(p\mu + p\sigma - \mu - \sigma)}{\beta(\delta + \mu)(\mu + \sigma + \tau + \omega)} \\
V^* &= \frac{pN(\beta\delta - \delta\mu - \mu^2)}{\beta(\delta + \mu)}
\end{aligned}$$

The Jacobian at the IFE this is:

$$J|_{IFE} = \begin{bmatrix} -\mu & 0 & -\beta & -\beta & -\beta \\ 0 & -\delta - \mu & \beta & \beta & \beta \\ 0 & \delta p & -\mu & 0 & 0 \\ 0 & \delta(1-p) & 0 & -\tau - \omega - \mu & \sigma \\ 0 & 0 & 0 & \tau + \omega & -\mu - \sigma \end{bmatrix}$$

The eigenvalues of this are:

$$\begin{aligned}
\lambda_1 &= -\tau - \sigma - \omega - \mu \\
\lambda_2 &= -\mu \\
\lambda_3 &= -\mu \\
\lambda_4 &= \frac{\delta}{2} - \mu + \frac{\sqrt{\delta(\delta + 4\beta)}}{2} \\
\lambda_5 &= -\frac{\delta}{2} - \mu - \frac{\sqrt{\delta(\delta + 4\beta)}}{2}
\end{aligned}$$

$\lambda_1, \lambda_2, \lambda_3$ and λ_5 are all clearly negative since all our parameters are positive. Thus the immigrant equilibrium is stable when $\lambda_4 < 0$, which occurs when $\frac{\beta\delta}{\mu(\mu+\delta)} < 1$.

After September eleventh the rate of deportation increased (thus increasing ω). Additionally, τ decreased, due to a fear of inability to return to the United States after leaving voluntarily. σ as well decreased after September eleventh due to increased border patrol. With constant parameters, our equilibria looks like 12

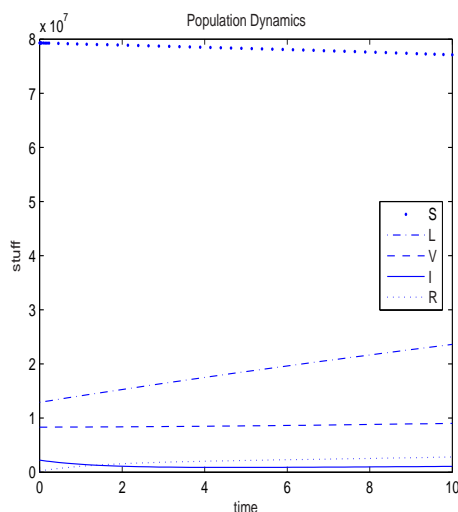


Figure 12: 9/11 Model Equilibria with no time dependence

We reanalyze the system assuming time dependent parameters (time=5 is September 11). We look at the time dependant parameters and at the dynamics in Figure 13.

In figure 14 we look at a side by side comparison of the model with time dependant parameters to time independant parameters (ignoring the S class for the sake of scale) As we can see, the time dependent parameters substantially changed the dynamics of the system—with the new time dependance, the R class quickly becomes larger than the I class.

Conclusions

In this paper, we have looked at three different and distinct models. In our first model, we looked at a caricature of Mexican immigration, diluting everything down to it's simplest. Looking at the stability of the immigrant free equilibrium, we decided it was always unstable, and thus that immigration would always persist. In our second model, we consider the effect of the quota on Mexican immigration. While we first thought that increasing the quota would help to eliminate illegal immigration by allowing all those that wanted to come into the United States to do so, we quickly discovered that increasing the quota increased all types of immigration. Our final model looked at the effect of the Patriot Act on immigration, and

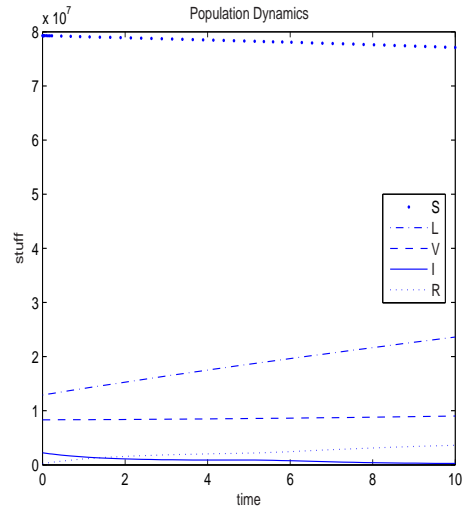
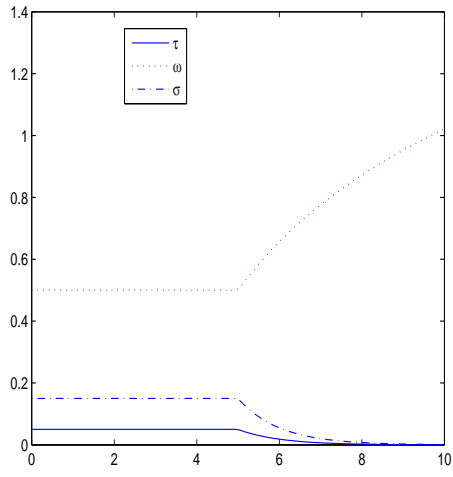


Figure 13: 9/11 Model Equilibria with time dependence

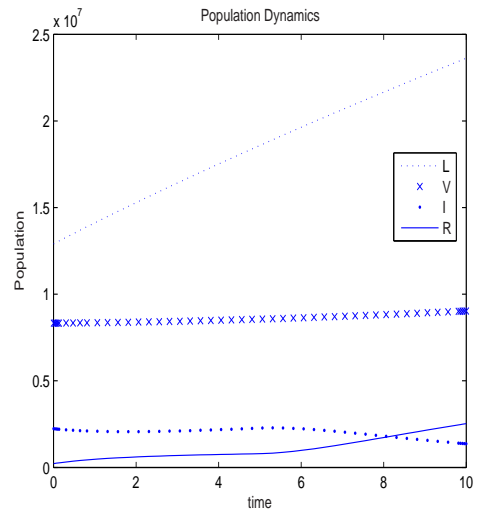
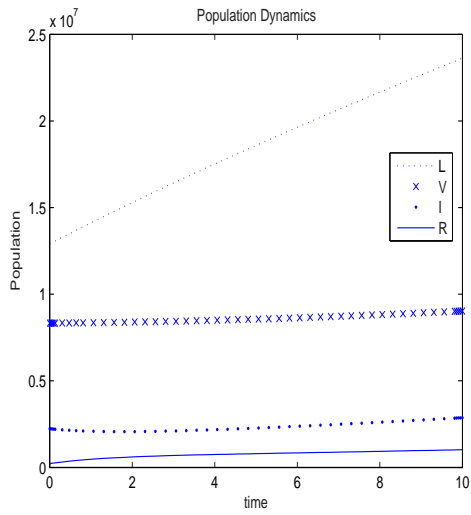


Figure 14: Dynamics with parameters time independent (left) and time dependent (right)

we conclude that the Patriot Act does, in fact, help to reduce the number of illegal immigrants in the country.

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Appendix

Simple Model

Rewriting R_0 as $\delta_2 = f(\delta_1)$

$$R = \frac{\beta}{\mu + \delta_1} + \frac{\delta_1}{\mu + \delta_1} \left(\frac{\delta_2}{\mu + \delta_2} + \frac{\beta\rho}{\mu + \delta_2} \right) = 1$$

$$\frac{\beta(\mu + \delta_2) + \delta_1(\delta_2 + \beta\rho)}{(\mu + \delta_1)(\mu + \delta_2)} = 1$$

$$\beta(\mu + \delta_2) + \delta_1(\delta_2 + \beta\rho) = (\mu + \delta_1)(\mu + \delta_2)$$

$$\beta\mu + \beta\delta_2 + \delta_1\delta_2 + \delta_1\beta\rho = \mu^2 + \delta_2\mu + \delta_1\mu + \delta_1\delta_2$$

$$\beta\mu + \beta\delta_2 + \delta_1\beta\rho = \mu^2 + \delta_2\mu + \delta_1\mu$$

$$\mu(\beta - \mu) + \delta_1(\beta\rho - \mu) + \delta_2(\beta - \mu) = 0$$

$$\delta_2(\beta - \mu) = -\mu(\beta - \mu) - \delta_1(\beta\rho - \mu)$$

$$\delta_2 = \frac{-\mu(\beta - \mu) - \delta_1(\beta\rho - \mu)}{\beta - \mu}$$

$$\delta_2 = -\mu - \frac{\delta_1(\beta\rho - \mu)}{(\beta - \mu)}$$

$$\delta_2 = \frac{\delta_1(\mu - \beta\rho)}{(\beta - \mu)} - \mu$$

Intercepts: If $\delta_1 = 0$, then $\delta_2 = -\mu$.

If $\delta_2 = 0$, then $\delta_1 = \frac{\mu(\beta - \mu)}{(\mu - \beta\rho)}$.

Quota Model Case One

Stability of the IFE

Stable when:

$$-\mu - \frac{\delta}{2} + \frac{1}{2}\sqrt{(\delta^2 + 4\beta\delta)} < 0$$

$$\sqrt{(\delta^2 + 4\beta\delta)} < 2\mu + \delta$$

$$\begin{aligned}
\delta^2 + 4\beta\delta &< (2\mu + \delta)^2 \\
\delta^2 + 4\beta\delta &< 4\mu^2 + 4\delta\mu + \delta^2 \\
\beta\delta &< \mu^2 + \delta\mu \\
\frac{\beta\delta}{\mu^2 + \delta\mu} &< 1 \\
\frac{\beta}{\mu} \frac{\delta}{\mu + \delta} &< 1 \\
R_0 &< 1
\end{aligned}$$

Quota Model Case Two

Existence Criteria for IE

System:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta S \frac{V+I}{N} - \mu S \\
\frac{dL}{dt} &= \beta S \frac{V+I}{N} - (\mu + \delta)L + \omega I \\
\frac{dV}{dt} &= \varphi - \mu V \\
\frac{dI}{dt} &= \delta L - \varphi - (\omega + \alpha + \mu)I \\
\frac{dN}{dt} &= \Lambda - \mu N - \alpha I
\end{aligned}$$

The equilibria of this system occurs when

$$\begin{aligned}
0 &= \Lambda - \beta S^* \frac{V^*+I^*}{N^*} - \mu S^* && \longrightarrow S^* = \frac{\Lambda}{\mu} \frac{\Lambda - \alpha I^*}{\beta V^* + (\beta - \alpha)I^* + \Lambda} \\
0 &= \beta S^* \frac{V^*+I^*}{N^*} - (\mu + \delta)L^* + \omega I^* \\
0 &= \varphi - \mu V^* && \longrightarrow V^* = \frac{\varphi}{\mu} \\
0 &= \delta L^* - \varphi - (\omega + \alpha + \mu)I^* && \longrightarrow L^* = \frac{\varphi}{\delta} + \frac{(\mu + \alpha + \omega)I^*}{\delta} \\
0 &= \Lambda - \mu N^* - \alpha I^* && \longrightarrow N^* = \frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*
\end{aligned}$$

All equations are now in terms of I^* and V^* (but V^* is a constant). Rescale all variables so $x = \frac{X^*}{\mu}$

$$\frac{\mu}{\Lambda} N^* = \left(\frac{\mu}{\Lambda}\right) \left(\frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*\right) \rightarrow n = 1 - \frac{\alpha}{\mu} i$$

$$\frac{\mu}{\Lambda} S^* = \frac{\mu}{\Lambda} \left(\frac{\Lambda}{\mu} \frac{\Lambda - \alpha I^*}{\beta V^* + (\beta - \alpha)I^* + \Lambda}\right) \rightarrow s = \left(\frac{\mu}{\Lambda}\right) \left(\frac{1}{\lambda}\right) \left(\frac{\Lambda - \alpha I^*}{\beta V^* + (\beta - \alpha)I^* + \Lambda}\right) \rightarrow s = \frac{\mu - \alpha i}{\beta v + (\beta - \alpha)i + \mu}$$

$$\frac{\mu}{\lambda} V^* = \frac{\mu}{\lambda} \frac{\varphi}{\mu} \rightarrow v = \frac{\varphi}{\lambda}$$

$$\frac{\mu}{\lambda} L^* = \left(\frac{\mu}{\Lambda}\right) \left(\frac{\varphi}{\delta} + \frac{I^*(\mu + \alpha + \omega)}{\delta}\right) \rightarrow l = \frac{\varphi}{\lambda} \frac{\mu}{\delta} + \frac{i(\mu + \alpha + \omega)}{\delta} \rightarrow l = \frac{\mu}{\delta} v + \frac{(\mu + \alpha + \omega)i}{\delta}$$

Since $S+L+I+V=N$, $S^* + L^* + I^* + V^* = N^*$, so $s + l + i + v = n$.

$$\frac{\mu - \alpha i}{\mu + \beta v + (\beta - \alpha)i} + \frac{\mu}{\delta} v + \frac{(\mu + \alpha + \omega)i}{\delta} + i + v = 1 - \frac{\alpha}{\mu} i$$

$$\frac{\mu - \alpha i}{\mu + \beta v + (\beta - \alpha)i} + \frac{\mu}{\delta} v + \frac{(\mu + \alpha + \omega)i}{\delta} + i + v - 1 + \frac{\alpha}{\mu} i = 0$$

$$\mu - \alpha i + (\mu + \beta v + (\beta - \alpha)i) \left(\frac{\mu}{\delta} v + \frac{(\mu + \alpha + \omega)i}{\delta} + i + v + 1 - \frac{\alpha}{\mu} i \right) = 0$$

$$\mu - \alpha i + (\mu + \beta v + (\beta - \alpha)i) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right) i + \left(\frac{\mu}{\delta} v - 1 \right) = 0$$

$$\left[(\beta - \alpha) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right) \right] i^2 + \left[-\beta + (\mu + \beta v) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right) + (\beta - \alpha) \left(\frac{\mu + \delta}{\delta} v \right) \right] i + \left[-\beta v + (\mu + \beta v) \left(\frac{\mu + \delta}{\delta} v \right) \right] = 0$$

$$(\beta - \alpha) \left(\frac{\mu^2 + \mu\alpha + \mu\omega + \delta\mu + \delta\alpha}{\delta\alpha} \right)$$

which simplifies to

$$(\beta - \alpha) \left(\frac{(\mu + \alpha)(\mu + \delta) + (\mu\omega) + \mu\omega}{\delta\mu} \right)$$

dividing through by the leading coefficient:

$$\frac{-\beta v + (\mu + \beta v) \left(\frac{\mu + \delta}{\delta} v \right)}{(\beta - \alpha) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right)}$$

$$\frac{\beta v}{\beta - \alpha} \left(\frac{-1 + \left(\frac{\mu}{\beta} + v \right) \left(\frac{\mu + \delta}{\delta} \right)}{(\mu + \alpha) \left(\frac{\mu + \delta}{\delta} \right) + \mu\omega} \right)$$

$$\frac{\beta v}{\beta - \alpha} \left(\frac{\left(\frac{\mu}{\beta} + v \right) - \frac{\mu + \delta}{\delta}}{(\mu + \alpha) \left(\frac{\mu + \delta}{\delta} \right) + \mu\omega} \frac{\delta}{\mu + \delta} \right) = v \frac{\beta}{\beta - \alpha} \left(\frac{\left(\frac{\mu}{\beta} + v \right) - \frac{\mu + \delta}{\delta}}{\frac{\mu + \alpha}{\omega} + \frac{\mu}{\mu + \delta}} \right)$$

$$\frac{-\beta + (\mu + \beta v) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right) + (\beta - \alpha) \left(\frac{\mu + \delta}{\delta} v \right)}{(\beta - \alpha) \left(\frac{\mu + \alpha + \omega}{\delta} + 1 + \frac{\alpha}{\mu} \right)}$$

$$\frac{-\beta}{(\beta - \alpha) \left(\frac{\mu + \alpha}{\delta\omega} (\mu + \delta) + \mu\omega \right)} + \frac{\mu + \beta v}{\beta - \alpha} + \frac{\frac{\mu + \delta}{\delta} v}{\frac{(\mu + \alpha)(\mu + \delta) + \mu\omega}{\delta\mu}}$$

note that $\frac{a}{bc} = \frac{\frac{a}{b}}{c}$, so

$$\frac{\mu + \beta v}{\beta - \alpha} + \frac{\frac{\mu + \delta}{\delta} v - \frac{\beta}{\beta - \alpha}}{\frac{(\mu + \alpha)(\mu + \delta) + \mu\omega}{\delta\omega}}$$

$$\frac{\beta}{\beta - \alpha} \left(\frac{\mu}{\beta} + v \right) + \frac{\frac{\mu + \delta}{\delta} v - \frac{\beta}{\beta - \alpha}}{\frac{(\mu + \alpha)(\mu + \delta) + \mu\omega}{\delta\omega}}$$

$$\frac{\beta}{\beta - \alpha} \left(\frac{\mu}{\beta} + v \right) + \frac{v - \frac{\beta}{\beta - \alpha} \frac{\delta}{\mu + \delta}}{\frac{\mu + \alpha}{\mu} + \frac{\omega}{\mu + \delta}}$$

$$i^2 + \left[\frac{\beta}{\beta - \alpha} \left(\frac{\mu}{\beta} + v \right) + \frac{v - \frac{\beta}{\beta - \alpha} \frac{\delta}{\mu + \delta}}{\frac{\mu + \alpha}{\mu} + \frac{\omega}{\mu + \delta}} \right] i = 0$$

Let $r = \frac{\delta}{\mu + \delta}$, $b = \frac{\beta - \alpha}{\beta}$, $m = \left(\frac{\mu + \alpha}{\mu} + \frac{\omega}{\mu + \delta} \right)^{-1}$, $d = \frac{\mu}{\beta}$ which gives us $0 < r, b, m < 1$ (if $\beta > \alpha$)

$$i^2 + \left(\frac{d+v}{b} + \left(v - \frac{r}{b} \right) m \right) i + v \frac{1}{b} m (d + v - r) = 0$$

Let $g(i) = i^2 + (\frac{d+v}{b} + (v - \frac{r}{b})m)i + v\frac{m}{b}(d + v - r)$
 $A = 1, B = mv - \frac{rm}{b} + \frac{d+v}{b}, C = v\frac{m}{b}(d + v - r)$

The roots of $g(i)$ are places where the value of i makes the system at an equilibrium.
 g will have 2 real roots when $B^2 - 4C$ is positive or $(mv - \frac{rm}{b} + \frac{d+v}{b})^2 - 4v\frac{m}{b}(d + v - r) > 0$

$$(mv)^2 + (\frac{rm}{b})^2 + (\frac{d+v}{b})^2 - 2mv\frac{rm}{b} + 2mv\frac{d+v}{b} - 2\frac{rm}{b}\frac{d+v}{b} - 4v\frac{m}{b}(d + v) + 4v\frac{m}{b}r > 0$$

$$(mv)^2 + (\frac{rm}{b})^2 + (\frac{d+v}{b})^2 - 2mv\frac{rm}{b} + 2mv\frac{d+v}{b} - 2\frac{rm}{b}\frac{d+v}{b} - 4v\frac{m}{b}(d+v) + 4(1-m)v\frac{rm}{b} + 4vm\frac{rm}{b} > 0$$

$$(mv)^2 + (\frac{rm}{b})^2 + (\frac{d+v}{b})^2 + 2mv\frac{rm}{b} - 2mv\frac{d+v}{b} - 2\frac{rm}{b}\frac{d+v}{b} + 4v(1-m)\frac{rm}{b} > 0$$

$(mv - \frac{d+v}{b} + \frac{rm}{b})^2 + 4mv\frac{rm}{b} > 0$, which is always true so g always has 2 real roots.

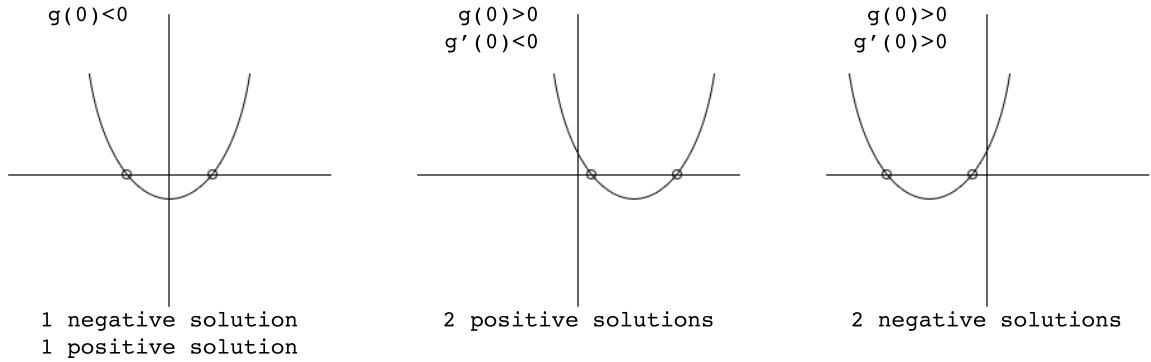
To explore these roots, we need to look at $g(0)$ and $g'(0)$:

$$g(0) = C = \frac{mv}{b}(v + d - r) > 0 \iff r < v + d$$

$$g'(0) = B = mv + \frac{v+d-rm}{b} > 0 \iff r < \frac{v(1+bm)+d}{m}$$

If $i > 0$, then the expressions for s, l, v and n are also positive as long as $i < \frac{\mu}{\alpha}$ (l is always positive for all $i \geq 0$, and v is independent of i , expressions for s and n are only positive if $i < \frac{\mu}{\alpha}$). Thus, the solutions of interest are those in $(0, \frac{\mu}{\alpha})$.

3 cases:



Looking at the right bound of $g(\frac{\mu}{\alpha}) > 0$

$$g(\frac{\mu}{\alpha}) = (\frac{\mu}{\alpha})^2 + (\frac{d+v}{b} + (v - \frac{r}{b})m)(\frac{\mu}{\alpha}) + v\frac{m}{b}(d + v - r)$$

$$g(\frac{\mu}{\alpha}) = (\frac{\mu}{\alpha})^2 + (\frac{d+v}{b} + (v - \frac{r}{b})m)(\frac{\mu}{\alpha}) + v(\frac{m}{b})(d + v - r) > 0$$

$$(\frac{\mu}{\alpha})^2 + \frac{d+v}{b}\frac{\mu}{\alpha} + vm\frac{\mu}{\alpha} - \frac{m}{b}\frac{\mu}{\alpha}r + \frac{vm(d+v)}{b} - \frac{vm}{b}r > 0$$

$$(\frac{\mu}{\alpha})^2 + (\frac{d+v}{b})(\frac{\mu}{\alpha} + vm) + vm(\frac{\mu}{\alpha}) > \frac{m}{b}(\frac{\mu}{\alpha} + v)r$$

$$\frac{(\frac{\mu}{\alpha})^2 + (\frac{d+v}{b})(\frac{\mu}{\alpha} + vm) + vm(\frac{\mu}{\alpha})}{\frac{m}{b}(\frac{\mu}{\alpha} + v)} > r$$

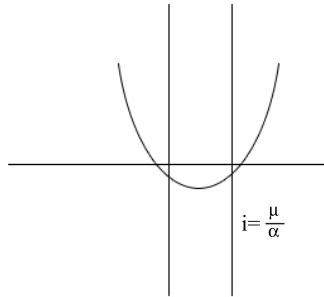
$$\frac{\frac{b}{m}((\frac{\mu}{\alpha})^2 + \frac{d+v}{b}m(\frac{\mu}{\alpha} + v) + vm\frac{\mu}{\alpha})}{\frac{\mu}{\alpha} + v} > r$$

$$\frac{\frac{b}{m}\frac{\mu^2}{\alpha^2} + (d+v)(\frac{\mu}{m} + v) + bv\frac{\mu}{\alpha}}{\frac{\mu}{\alpha} + v} > r$$

$$\frac{(\frac{\mu}{\alpha} + v)(d + v + b\frac{\mu}{\alpha})}{\frac{\mu}{\alpha} + v} > r$$

if $g(\frac{\mu}{\alpha}) < 0$, $r > r_3$, thus $g(0) < 0$

thus, $r > r_3$ means no positive solution for i .

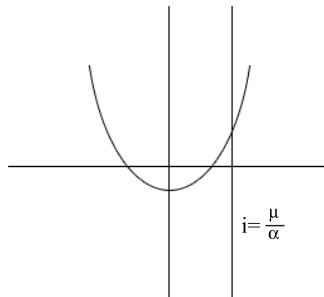


if $r < r_1$, then $g(0) > 0$ if $r < r_1$, since $r_1 < r_2$, $r < r_2$ so $g'(0) > 0$

This gives us 2 negative solutions, so for $r < r_1$, we have no solutions.

To have 2 positive solutions we need $g(0) > 0$, or $r < r_1$, and $g'(0) < 0$, or $r > r_2$ since $r_1 < r_2$, this is impossible, so we cannot have 2 positive solutions.

Thus, for $r_1 < r < r_3$, we have $g(0) < 0$ and $g(\frac{\mu}{\alpha}) > 0$ or this gives us 1 positive solution.



Thus, there is only 1 IE and it exists when $r_1 < r < r_3$.

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