An Epidemiological Approach to the Spread of Political Third Parties

Karl Calderon[1], Clara Orbe[2], Azra Panjwani[3], Daniel M. Romero[4], Christopher Kribs-Zaleta[5], Karen Ríos-Soto[6]

[1] University of Arizona,

Department of Mathematics, Tucson, AZ

[2] Brown University,

Department of Applied Mathematics, Providence, RI

[3] University of California,

Department of Mathematics, Berkeley, Berkeley, CA

[4] Arizona State University,

Department of Mathematics and Statistics, Tempe, AZ

[5] University of Texas, Arlington,

Department of Mathematics, Arlington, TX

[6] Cornell University,

Department of Biometrics, Ithaca, NY

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Abstract

Third political parties are influential in shaping American politics. In this work we study the spread of third parties ideologies in a voting population where we assume that party members are more influential in recruiting new third party voters than non-member third party voters. The study is conducted using an epidemiological model with nonlinear ordinary differential equations as applied to a case study, the Green Party. Through the analysis of our system we obtain the party-free and member-free equilibria as well as two endemic equilibria. We identify two threshold parameters in our model that describe the different possible scenarios for the political parties and their spread. Our system produces a backward bifurcation that helps identify conditions under which a third party can thrive. We perform a sensitivity analysis to the threshold conditions in order to isolate those parameters to which our model is most sensitive. We explore all results through deterministic simulations and refer to data from the Green Party in the state of Pennsylvania as a case study.

1 Introduction

The 2000 United States presidential election was for many a testimony to the impact of third parties in a traditionally bipartisan government. Ralph Nader, the presidential candidate for the Green Party, won 2% of the popular vote, a percentage that many attribute to the defeat of Democratic candidate Al Gore [18]. The Green Party captured a seemingly insignificant number of votes relative to majority percentages, yet its presence in the election ultimately served to shape American politics for the years following. This incident demonstrates how third parties, often emerging as grassroots movements (i.e., movements at the local level rather than at the center of major political activity), can ultimately impact at the national level, hence prompting the need to study their emergence and spread within a voting population.

Third parties are defined as political parties operating along with two major parties in a bipartisan system over a limited period of time (where we define a limited period of time as a range of a few years). For the purposes of this paper we apply this definition to all minor parties. Traditionally, third parties have served as venues of political dissent for voting individuals dissatisfied with the major candidates in an election. They often tackle specific issues otherwise ignored by major political parties, thus relinquishing popular support nationwide. As Supreme Court Justice Black wrote in 1968, "History has amply proved the virtue of political activity by minority, dissident groups, which innumerable times have been in the vanguard of democratic thought and whose programs were ultimately accepted" [22]. Hence, while third parties rarely capture the majority vote, their agendas, often incorporated into major party platforms, are significant nonetheless.

Given the potential relevance of third parties to national politics, we strive to qualitatively and numerically study the dynamics of the emergence and spread of third parties on a local level where growth is measured in terms of the number of third party voters and members. We restrict our study to a local level because third parties usually originate in a small group and, via a 'bottom-up' method of diffusion, spread within a population by acquiring local official positions and then expanding to higher levels of government [16]. We use an epidemiological paradigm to translate third party emergence from a political phenomenon to a mathematical one where we assume that third parties grow in a similar manner as epidemics in a population. We take this approach following in the steps of previous quantitative studies that model social issues via such methods [7].

While our model is designed to pertain to all third parties, we consider the Green Party as a case study. Although formally united under the Association of State Green Parties in 1996 (then the nationalized Green Party of the United States in

2001), state-based green parties have thrived at the local level since 1984 where they seek to continue the Populist's fight for citizen empowerment and progressive reform via a set of ten core values (See Appendix C)[14]. Our particular study focuses on the Green Party of Pennsylvania where we observe the party's growth from 2001 to 2005. In comparing deterministic simulations of our model to particular data, we consider a short time frame so that we can assume that social structure within the state in question does not change drastically, a necessary condition for assuming voting population heterogeneity.

We apply a system of nonlinear differential equations to a population of voting individuals that, according to certain demographic factors, are heterogeneously susceptible to third party ideology. We consider susceptible movement into voting and member compartments and possible regressions back from the third party voting phase into the susceptible class. In order to facilitate analysis of the two-track model, we initially consider a simplified version that does away with voting population heterogeneity and then explore analysis for the more complex system. We solve for equilibria, identify those conditions for the instability of both the party-free equilibrium (analogous to a disease-free equilibrium in an epidemic) and member-free equilibrium, identify two thresholds, and explore the resulting backward bifurcation at which our endemic equilibrium associated with party survival exists and is stable. Via sensitivity analysis we identify those parameters that most impact the recruitment of third party voters and rewrite these quantitative findings in political terms. Additionally, we run deterministic simulations of the model for conditions that guarantee the existence of all four expected equilibria.

We organize our paper as follows: Section 2 introduces the two-track model, Section 3 reduces the complex version into the one-track model on which we perform analysis (i.e., equilibria analysis, bifurcation plots, deterministic simulations), Section 4 references an individual case study and applies data for parameter estimation, Section 5 performs a similar yet more limited analysis of the complex model and Section 6 concludes with recommendations and advice to help third parties better strategize voter recruitment.

1.1 A Population Model for the Spread of a Third Party

Our model considers a population of all voters, N, divided into two classes or sub-populations whose susceptibility to third party ideology is based on the following demographic factors: education, socioeconomic status, race, gender, age, political orientation and professional occupation. We assume that individuals do not move from one susceptible class to the other. See Assumption 3 in Section 2.3 (hereon refer to all assumptions in Section 2.3). Inherently, certain demographic factors, labelled as high affinity, make an individual more likely to subscribe to a third party's ideology that, as detailed in Assumption 4, targets a more specific audience than alternative majority agendas. For this reason we consider population heterogeneity vital to this study. When an individual enters the voting system that person, due to his/her characteristic demographic factors, is more statistically inclined to vote a certain way. For example, a progressive environmental activist is statistically more likely to agree and vote for the Green Party agenda, that stresses communal based economics, local government, and gender and racial equity, than a conservative corporate executive whose economic philosophy directly conflicts with that of the Green Party.

We apply the following method of dividing the entering voting population into two susceptible classes: if an individual has more high affinity factors than low affinity factors then that person directly enters the high affinity class and similarly for low affinity susceptibles. We define high affinity factors as features of the individual based on his/her demographic profile that make him/her more inclined to vote for the third party; conversely, low affinity factors make the individual less statistically likely to subscribe to the party's platform. Each party has its own agenda that appeals to certain sectors of the voting population. Hence different parties target voting populations that are more inclined to subscribe to their ideology. While one party, for example, may target individuals from a certain educational background that we, in our model, label as high affinity and that other parties may overlook, all parties nonetheless recognize that education factors into an individual's likelihood to support or refute that party's platform. It is true that individuals from varied backgrounds comprise the main parties yet, when dealing with the specific agendas of third parties that do not strive to sway the majority vote, we assume that third parties appeal to individuals of certain demographic backgrounds more than others. See Assumption 4. Therefore, we account for the aforementioned standard set of demographic factors that parties look at when spreading their ideologies. In our paper we apply our model to an individual case study of the Green Party of Pennsylvania; however, the same methodology of distinguishing susceptibles can be applied to all third parties.

All voters enter the voting system either to the low affinity, L, or high affinity, H, susceptible class. Our Susceptible/Infected-based model then considers two susceptible and three infected classes, V_H, V_L, and M, third party voters from the high affinity class, third party voters from the low affinity class and party members respectively. We define party members as voters of the third party who pay dues to the party; often such members officiate, volunteer and actively campaign for voter recruitment. We apply epidemiological terminology, specifically 'infectious', when referring to susceptibles who transition to the voting and, possibly, member classes. In biological terms, V_H and V_L correspond to voters of a lower degree of infection and individuals of the M class are voters infected to a higher degree. Once an individual is susceptible he/she can become 'infected' (either V_H or V_L) through direct contact with the V_H, V_L, and M classes. We do not consider a linear term weighing the influence of media coverage from the third party (i.e., secondary contact factors) in the forward transition from both susceptible classes to third party voting classes. See Assumption 5. Instead, we focus on the nonlinear terms considering the effects of voters from the V_H, V_L, and M classes where voters from V_H and V_L bear an equal influence β_1 (from H to V_H) and β_2 (from L to V_L) in third party voter recruitment; members from M influence susceptibles at a rate proportional to voter influence, embodied in the multiplicative factors $\alpha\beta_1$ and $\alpha\beta_2$. See Assumption 6.

We consider the transition back from the third party voting to the susceptible classes that involves the linear terms, $\epsilon_1 V_H$ and $\epsilon_2 V_L$, and nonlinear terms, $\phi_1(\tau H + L)\frac{V_H}{N}$ and $\phi_2(\tau L + H)\frac{V_L}{N}$, contributions by secondary contacts with the opposition (i.e., media from well-funded majority voters) and direct contact with the susceptible classes respectively. See Assumption 7. Similar to α in the forward transition, we designate τ as the augmentation factor. Therefore, in regressing back from V_H to H, τH represents the greater influence that H individuals exert on voters from V_H than susceptibles from L, the lower affinity class (similar reasoning applies to the V_L to L transition). See Assumption 10.

Once voting for the third party, individuals can become party members. They enter this higher state of infection via the nonlinear terms $\gamma V_H \frac{M}{N}$ and $\gamma V_L \frac{M}{N}$, where we only consider the effects of party membership on bringing about this transition embodied in the general parameter γ. See Assumption 5. Given that we are studying the spread of the party we assume that party members stay in M and do not regress to other classes. See Assumption 9.

Finally, we consider natural exits from all classes as a result of death or moving. The sum of the equations of the model, for both the complex and simple versions, $\frac{dN}{dt} = 0$, verifies that our population stays constant, a safe assumption by which the number of people entering the voting system (i.e., coming of age, moving in) counterbalances the number of people leaving the system (i.e., dying, moving out).

2 The Two-Track Model

We introduce a two-track model to study the dynamics between a heterogeneously mixed population of susceptible voters, third party voters, and party members.

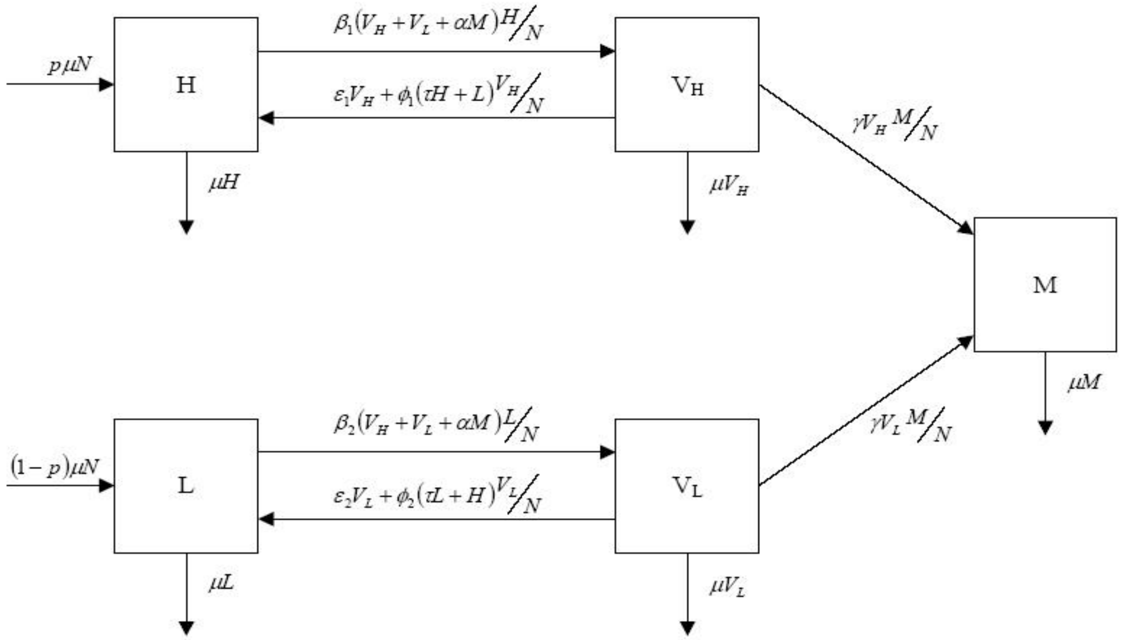

Figure 1: The two-track model

2.1 Equations

We apply the following set of ordinary differential equations to model voting dynamics.

$$\frac{dH}{dt} = p\mu N + \epsilon_1 V_H + \phi_1(\tau H + L)\frac{V_H}{N} - \beta_1(V_H + V_L + \alpha M)\frac{H}{N} - \mu H, \tag{1}$$

$$\frac{dL}{dt} = (1-p)\mu N + \epsilon_2 V_L + \phi_2(\tau L + H)\frac{V_L}{N} - \beta_2(V_H + V_L + \alpha M)\frac{L}{N} - \mu L, \tag{2}$$

$$\frac{dV_H}{dt} = \beta_1(V_H + V_L + \alpha M)\frac{H}{N} - \epsilon_1 V_H - \phi_1(\tau H + L)\frac{V_H}{N} - \frac{\gamma M V_H}{N} - \mu V_H, \tag{3}$$

$$\frac{dV_L}{dt} = \beta_2(V_H + V_L + \alpha M)\frac{L}{N} - \epsilon_2 V_L - \phi_2(\tau L + H)\frac{V_L}{N} - \frac{\gamma M V_L}{N} - \mu V_L, \tag{4}$$

$$\frac{dM}{dt} = \frac{\gamma M V_H}{N} + \frac{\gamma M V_L}{N} - \mu M, \tag{5}$$

$$N = H + L + V_H + V_L + M. \tag{6}$$

where N is the total population. Adding equations (1), (2), (3), (4) and (5) we obtain $\frac{dN}{dt} = 0$, showing that the total population N is constant over time.

2.2 Compartments and Parameters

Refer to Table 1.

2.3 Model and Background Assumptions

We preface our description with a list of assumptions essential to our model:

(1) A party exists only if it has members where we define members as those who pay dues, volunteer, and preside over party affairs.

(2) We assume that our population is a heterogeneous mix of individuals who belong to different backgrounds according to certain demographic factors. See model description for more detail.

(3) We limit our model to tracing the *expansion* of the third party; hence, we refer to a shorter time period over which we assume social structure remains constant and individuals do not travel directly between H and L.

(4) In addition to being more specific than major party agendas (i.e., more specific in their goals and less geared to moderacy), third party platforms tend to be more consistent over time. Third parties are not pressured to constantly adjust

Table of Parameters and Compartments	
H	high affinity susceptibles (i.e. voters highly susceptible to third party ideology)
L	low affinity susceptibles (i.e. voters barely susceptible to third party ideology)
V_H	third party voting individuals deriving from H
V_L	third party voter individuals deriving from S
M	third party members (i.e. party officials, donors, volunteers)
p	proportion of the voting population N entering H
β_1	peer driven recruitment rate of H into V_H by individuals in V_H, V_L and M
α	factor by which the recruitment rate of H and L into V_H and V_L by individuals in M exceeds the recruitment rate by individuals in V_H and V_L
ϵ_1	linear recruitment rate of V_H back into H via secondary contacts (i.e., media and campaigning from opposing parties)
ϕ_1	recruitment rate of V_H into H by direct contact with individuals in the opposition classes (i.e., individuals in H and L)
β_2	peer driven recruitment rate of L into V_L by individuals in V_L, V_H, and M (analogous to β_1).
τ	factor by which the recruitment rate of V_H and V_L by members of the same susceptible class exceeds the recruitment rate by susceptibles of the other class
ϵ_2	linear recruitment rate of V_L back into L via secondary contacts (i.e., media and campaigning from opposing parties)
ϕ_2	recruitment rate of V_L into L by direct contact with individuals in the opposition classes (i.e., individuals in H and L)
γ	recruitment rate of V_H and V_L into M by individuals in M
μ	rate at which individuals enter or leave the voting system

Table 1: Compartments and parameters of the two-track model.

to the shifting demands of the populace since they do not seek the majority vote. Consequently, they do not target the majority voting population.

(5) We assume that third parties, due to a lack of funding and resulting lack of media exposure to the general population, spread mainly via primary contacts among susceptibles, H and L, third party voters, V_H and V_L, and members, M, at the rates β and $\alpha\beta$ respectively. We define such direct interaction as personal meetings, phone conversations, and personally addressed emails.

(6) We assume that third party voters from V_H and V_L equally influence H and L into becoming third party voters. However, since party members correspond to the higher degree of 'infection,' members have a greater effect in voter recruitment, embodied in the augmentation parameter α.

(7) We consider both primary contacts and secondary contacts in the regression of third party voters to the susceptible class via the ϕ (nonlinear) and ϵ (linear) terms. Compared to primary contacts, described in assumption (5), secondary contacts include mass emails, media, and circulating literature.

(8) We assume that all other parties exert equal influence in discouraging third party voting.

(9) Party members do not resign their memberships. We reason that once an individual feels strongly enough to join a party he/she retains his/her loyalty to the party; the only way a person can exit M is by leaving the voting system.

(10) Voters from a certain susceptibility class (with its own set of demographic factors) address issues that usually appeal more to third party voters deriving from the same class. Therefore, susceptibles from H bear a greater influence, embodied in the parameter τ, in recruiting V_H voters back into the susceptible class than L susceptibles.

3 Analysis

3.1 A Simplification: The Decoupled System

In order to facilitate analysis we assume a homogeneous susceptible population and reduce the two-track model into one susceptible class, S, and two infected classes, third party voters, V, and party members, M, respectively. In the one-track model we omit unnecessary parameters from the heterogeneous version.

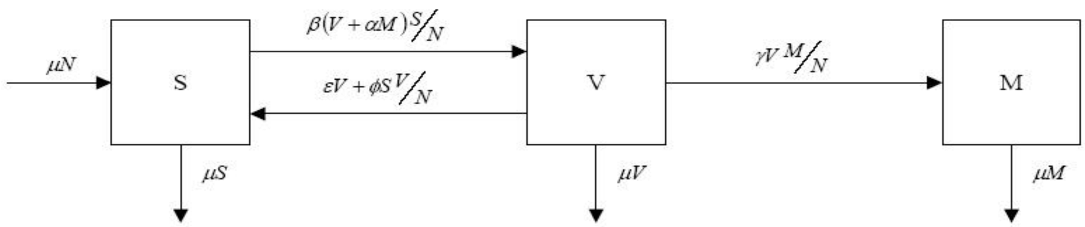

Figure 2: The one-track model

3.2 Equations

In this case the model reduces to the following two-dimensional model since N is constant:

$$\frac{dS}{dt} = \mu N + \epsilon V + \phi S\frac{V}{N} - \beta(V+\alpha M)\frac{S}{N} - \mu S, \tag{7}$$

$$\frac{dV}{dt} = \beta(V+\alpha M)\frac{S}{N} - \epsilon V - \phi S\frac{V}{N} - \frac{\gamma MV}{N} - \mu V, \tag{8}$$

$$\frac{dM}{dt} = \frac{\gamma MV}{N} - \mu M, \tag{9}$$

$$N = S + V + M. \tag{10}$$

where N is the total population. Adding equations (7), (8), and (9) we obtain $\frac{dN}{dt} = 0$, showing that N is constant over time.

3.3 Compartments and Parameters

The S class comprises susceptible individuals (i.e., those people who vote, but do not vote for the third party). The V class comprises the third party voters and the M class has third party members (i.e., party officials, donors, volunteers). Refer to Table 2.

3.4 Equilibria Analysis

We approach our analysis by calculating equilibria and determining conditions of equilibria existence and stability.

β	peer driven recruitment rate of S into V by third party voters and members
ϵ	recruitment rate of V back into S via secondary contacts (i.e., media and campaigning from opposing parties)
ϕ	recruitment rate of V into S by direct contact with susceptibles
α	factor by which the recruitment rate of S into V by third party members exceeds the recruitment rate by individuals in V
γ	recruitment rate of V into M by third party members
μ	rate at which individuals enter or leave the voting system

Table 2: Parameters of the one-track model.

3.4.1 Reducing and Proportionalizing the System of Equations

The decoupled system contains three equations and, since our total population is constant, we can reduce our system to two dimensions via the substitution $S = N - V - M$ into equation (8). Therefore, $\frac{dV}{dt}$ becomes

$$\frac{dV}{dt} = \beta(V + \alpha M)\frac{N - V - M}{N} - \epsilon V - \phi(N - V - M)\frac{V}{N} - \frac{\gamma MV}{N} - \mu V. \qquad (11)$$

which allows us to ignore $\frac{dS}{dt}$ so that we now have a reduced system of equations.

As for proportionalizing the system (i.e., defining our variables as proportions of the total voting population N) let $s = \frac{S}{N}$, $v = \frac{V}{N}$ and $m = \frac{M}{N}$. Similarly, let $s^* = \frac{S^*}{N}$, $v^* = \frac{V^*}{N}$ and $m^* = \frac{M^*}{N}$ where (S^*, V^*, M^*) is an equilibrium point for the unreduced system. Dividing equation (10) through by N we get $1 = s + v + m$. Dividing equations (11) and (9) through by N and substituting in s, v and m, we find that:

$$\frac{dv}{dt} = \beta(v + \alpha m)(1 - v - m) - \epsilon v - \phi(1 - v - m)v - \gamma mv - \mu v, \qquad (12)$$

$$\frac{dm}{dt} = (\gamma v - \mu)m. \qquad (13)$$

In order to analyze stability we linearize the system and compute partial first derivatives with respect to each of the variables, v and m:

Jacobian for the Reduced System:

$J_2 =$

$$\begin{pmatrix} (\beta-\phi)(1-2v-m)-\alpha\beta m-(\mu+\epsilon)-\gamma m & \alpha\beta(1-v-2m)-(\beta-\phi)v-\gamma v \\ \gamma m & \gamma v-\mu \end{pmatrix}$$

3.4.2 E_1: Party-Free Equilibrium (PFE)

The party-free equilibrium (PFE) for the reduced system occurs at (0,0), the steady state achieved when the entire population resides in the S class (i.e., the third party has neither voters nor members and, by definition of party existence, does not exist). The PFE is essentially analogous to the disease-free equilibrium in biology and always exists as a possible outcome for the voting population.

Applying the above reduced Jacobian matrix to our PFE, (0,0), where we only consider the v and m terms, we determine PFE stability:

$$J_2(0,0) = \begin{pmatrix} (\beta-\phi)-(\mu+\epsilon) & \alpha\beta \\ 0 & -\mu \end{pmatrix}$$

The equilibrium point (0,0) will be locally asymptotically stable (LAS) if all the eigenvalues of the matrix are negative. Assuming $\mu > 0$, the eigenvalue $-\mu$ of the Jacobian is always negative, whereas the second eigenvalue $(\beta-\phi)-(\mu+\epsilon) < 0$ if and only if $\frac{(\beta-\phi)}{\mu+\epsilon} < 1$. Therefore, in order to meet this criterion for stability of the PFE we assign $R_1 = \frac{(\beta-\phi)}{\mu+\epsilon} < 1$.

We discuss the relevance of this threshold value in section 3.6, *Threshold Parameters R_1 and R_2.*

3.4.3 E_2: Member-Free Equilibrium (MFE)

This situation occurs when $M = 0$ but $V, S \neq 0$, (i.e, the voting population subdivides between susceptibles, S, and third party voters, V). While mathematically possible, this outcome is politically unrealistic given that voters cannot vote for a party that does not exist (we assume that party existence depends on the presence of an M class). See Assumption 1. For mathematical consistency, however, we consider the equilibrium point (s_2^*,v_2^*,m_2^*), where $m_2^* = 0$. Note that the subscript of the equilibria values refers to the equilibrium point in question (we distinguish between the endemic equilibria values ahead via the $\pm$ subscript). (We discuss the situation where $m \neq 0$ in section 3.5.4, *Endemic Equilibria*).

If $m_2^* = 0$, then $s_2^* + v_2^* = 1$ which, in turn, implies that $s_2^* = 1 - v_2^*$.

Rearranging equation (12) of the decoupled system together while substituting in $m_2^* = 0$ and v_2^* we have:

$$(\beta - \phi)v_2^{*2} + (\mu + \epsilon + \phi - \beta)v_2^* = 0 \tag{14}$$

This implies that either $v_2^* = 0$ or $(\beta - \phi)v_2^* = \beta - (\mu + \epsilon + \phi)$.

We consider the situation where $v_2^* \neq 0$, solve for v_2^* and simplify the results as follows:

$$v_2^* = 1 - \frac{\mu + \epsilon}{\beta - \phi} = 1 - \frac{1}{R_1}$$

where v_2^* retains political value only if $R_1 > 1$ or else $v_2^* < 0$ which is a contradiction given that $0 \leq 1$ and is a politically irrelevant number of individuals.

Similarly we solve for s_2^* and obtain $s_2^* = 1 - v_2^* = \frac{\mu+\epsilon}{\beta-\phi} = \frac{1}{R_1}$ which makes sense politically only if $\beta > \phi$ (i.e, $s_2 > 0$). From solving for s_2^* and v_2^* above, we express our member-free equilibrium as $E_2 = (\frac{\mu+\epsilon}{\beta-\phi}, 1 - \frac{\mu+\epsilon}{\beta-\phi}, 0)$.

Expressed in terms of R_1, the MFE is $(\frac{1}{R_1}, 1 - \frac{1}{R_1}, 0)$ and exists if and only if $R_1 > 1$, since ignoring this condition leads to an otherwise negative third party voting population. If $R_1 > 1$, then $\frac{1}{R_1} < 1$ (i.e., the entire population does not reside in the susceptible class). This implies that the population has moved out of S into the V and M classes. Since $M = 0$, the left over proportion (i.e., $1 - \frac{1}{R_1}$) resides in V.

The above situation makes mathematical sense but not political sense since parties, by our original assumption, do not exist without members and in this member-free case we deal with voters that vote for a non-existent party.

Regardless of the political likelihood of MFE existence, we consider its stability. Again, we apply the method of using the reduced system Jacobian matrix in determining the stability of the member-free equilibrium (where $v^* \neq 0$):

$$J_2(1 - \tfrac{\mu+\epsilon}{\beta-\phi}, 0) =$$

$$\begin{pmatrix} (\beta-\phi)(1 - 2(1 - \frac{\mu+\epsilon}{\beta-\phi})) - (\mu+\epsilon) & \alpha\beta(1 - (1 - \frac{\mu+\epsilon}{\beta-\phi})) - ((\beta-\phi)+\gamma)(1 - \frac{\mu+\epsilon}{\beta-\phi}) \\ 0 & \gamma(1 - \frac{\mu+\epsilon}{\beta-\phi}) - \mu \end{pmatrix}$$

which, expressed in terms of R_1 is

$$J_2(1-\frac{1}{R_1},0) = \begin{pmatrix} (\beta-\phi)(1-2(1-\frac{1}{R_1}))-(\mu+\epsilon) & \alpha\beta(\frac{1}{R_1})-(\beta-\phi+\gamma)(1-\frac{1}{R_1}) \\ 0 & \gamma(1-\frac{1}{R_1})-\mu \end{pmatrix}$$

The reduced system equilibrium point $(1-\frac{1}{R_1},0)$ is locally asymptotically stable if all the eigenvalues of the above matrix are negative. We know that, since $\beta > \phi$, one of the eigenvalues, $(\beta-\phi)(1-2(1-\frac{\mu+\epsilon}{\beta-\phi}))-(\mu+\epsilon) < 0$ if and only if

$$1-2\left(1-\frac{1}{R_1}\right) < \frac{1}{R_1}$$

$$1-\frac{1}{R_1} < 2\left(1-\frac{1}{R_1}\right)$$

$$1 < 2$$

which is always true so long as $R_1 > 1$.

The second eigenvalue of the Jacobian is $\gamma(1-\frac{1}{R_1})-\mu$. This eigenvalue is negative if and only if $\frac{\gamma}{\mu}(1-\frac{1}{R_1}) < 1$.

We define the left hand side of the inequality as $R_2 = \frac{\gamma}{\mu}(1-\frac{\mu+\epsilon}{\beta-\phi}) = \frac{\gamma}{\mu}(1-\frac{1}{R_1})$. Hence we have derived two threshold parameters R_1 and R_2 that determine equilibria stability depending on relative parameter values.

3.4.4 E_3 and E_4: Endemic Equilibria

In the event of the endemic equilibria, the voting population subdivides between susceptibles, S, third party voters, V, and members, M. We regard this as a successful state of coexistence and, given certain conditions, the point at which the party thrives. Since $\frac{dm}{dt} = (\gamma v^* - \mu)m^*$, $v^* = \frac{\mu}{\gamma}$ when $m^* \neq 0$. At this point we do not use a particular subscript in order to leave room open for the possibility of two endemic equilibria E_3 and E_4 (we will apply subscripts when dealing with each specific case). Also we impose the condition $\mu < \gamma$ so that m_2^* is not negative and $h^* = 1 - v^* - m^* = 1 - \frac{\mu}{\gamma} - m^*$ for the endemic equilibria.

We rearrange equation (16) and substitute in $v^* = \frac{\mu}{\gamma}$ in order to solve for m^*:

$$\frac{dv}{dt} = (\beta-\phi)\left(\frac{\mu}{\gamma}\right)\left(1-\frac{\mu}{\gamma}-m^*\right)+\alpha\beta m^*\left(1-\frac{\mu}{\gamma}-m^*\right)-(\mu+\epsilon+\gamma m^*)\left(\frac{\mu}{\gamma}\right) = 0$$

After simplification and dividing through by $\alpha\beta$ $\frac{dv}{dt}$ reduces to a quadratic expression

$f(m^*)$ in terms of m^*

$$f(m^*) = m^{*2}+\left[\left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma}-\left(1-\frac{\mu}{\gamma}\right)+\frac{\mu}{\alpha\beta}\right]m^*-\frac{\mu}{\gamma}\left[\frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right)-\frac{\mu+\epsilon}{\alpha\beta}\right]=0$$

We prove the coexistence of two positive endemic equilibria, where $m^* = m^*_{\pm}$ by showing that $f(m^*)$ has two solutions in $(0, 1)$ whenever $B < 0$, $C > 0$ and $B^2 - 4AC > 0$ are true, as defined below. (See appendix section A.1 for proof). From this point on we use the subscripts $\pm$ to distinguish the two endemic equilibria.

In order to show the coexistence of two member states, we solve for $m^*_{\pm}$ using the quadratic formula:

$$m^*_{\pm} =$$

$$\frac{1}{2}\left[-\left[\left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma}-\left(1-\frac{\mu}{\gamma}\right)+\frac{\mu}{\alpha\beta}\right]\pm\sqrt{\left[\left(-\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma}+\left(1-\frac{\mu}{\gamma}\right)-\frac{\mu}{\alpha\beta}\right]^2+4\left(\frac{\mu}{\gamma}\right)\left[\frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right)-\frac{\mu+\epsilon}{\alpha\beta}\right]}\right]$$

Where

$$A = 1,$$
$$B = \left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma}-\left(1-\frac{\mu}{\gamma}\right)+\frac{\mu}{\alpha\beta},$$
$$C = -\frac{\mu}{\gamma}\left[\frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right)-\frac{\mu+\epsilon}{\alpha\beta}\right].$$

After establishing conditions for the existence of two endemic equilibria, $E_3 = (1 - \frac{\gamma}{\mu} - m^*_+, \frac{\gamma}{\mu}, m^*_+)$ and $E_4 = (1 - \frac{\gamma}{\mu} - m^*_-, \frac{\gamma}{\mu}, m^*_-)$, we discuss equilibria stability.

The reduced Jacobian matrix for the endemic equilibria follows:

$$J_2(\tfrac{\mu}{\gamma}, m^*_{\pm}) =$$

$$\begin{pmatrix} (\beta-\phi)\left(1-2\frac{\mu}{\gamma}-m^*_{\pm}\right)-\alpha\beta m^*_{\pm}-(\mu+\epsilon)-\gamma m^*_{\pm} & \alpha\beta\left(1-\frac{\mu}{\gamma}-2m^*_{\pm}\right)-(\beta-\phi)\frac{\mu}{\gamma}-\mu \\ \gamma m^*_{\pm} & 0 \end{pmatrix}$$

and its characteristic equation is:

$$p(\lambda)=$$

$$\lambda^2+\left[-(\beta-\phi)\left(1-\frac{2\mu}{\gamma}-m^*_{\pm}\right)+\alpha\beta m^*_{\pm}+\mu+\epsilon+\gamma m^*_{\pm}\right]\lambda-\alpha m^*_{\pm}\left[\alpha\beta\left(1-\frac{\mu}{\gamma}-2m^*_{\pm}\right)-\frac{(\beta-\phi)\mu}{\gamma}-\mu\right]=0$$

The solutions of the characteristic equation are the eigenvalues of $J_2(\frac{\mu}{\gamma}, m^*_\pm)$, and the stability of the endemic equilibria depends on the signs of the real parts of the eigenvalues. If the real parts of both eigenvalues are negative then we conclude that the endemic equilibrium in question is locally asymptotically stable. Based on this criterion, we find that E_3 is always LAS when it exists while E_4 is never LAS when it exists. (See appendix A.2 for details).

3.4.5 Equilibria, a Summary

Table 3 summarizes and classifies the equilibria of the decoupled system.

Equilibria	Existence	Stability
E_1 Party-Free (1,0,0)	always exists	LAS if $R_1 = \frac{\beta-\phi}{\mu+\epsilon} < 1$
E_2 Member-Free $(\frac{1}{R_1}, 1-\frac{1}{R_1}, 0)$	exists $\leftrightarrow$ $R_1 > 1$	LAS $\leftrightarrow$ $R_2 = (\frac{\gamma}{\mu})(1-\frac{1}{R_1}) < 1$
E_3 Coexistence/Endemic $(1-\frac{\mu}{\gamma}-m^*_+, \frac{\mu}{\gamma}, m^*_+)$	exists $\leftrightarrow$(1) $R_2 > 1$ or (2)$R_2 < 1$**	LAS
E_4 Coexistence/Endemic $(1-\frac{\mu}{\gamma}-m^*_-, \frac{\mu}{\gamma}, m^*_-)$	exists $\leftrightarrow$ (1) $R_2 < 1$**	unstable

**AND conditions (a) and (b) are met

Table 3: Equilibria of one-track model.

Conditions and defined variables:

(a) $\frac{\mu}{\alpha\beta} + (\frac{\beta-\phi}{\alpha\beta}+1)\frac{\mu}{\gamma} < 1$

(b) $\frac{\mu}{\gamma} \le \frac{\mu}{\alpha\beta} + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - 2\sqrt{\frac{\mu}{\alpha\beta}\left(1+\frac{\epsilon}{\gamma}\right)} + 1$

(c) $m^*_\pm = -\frac{1}{2}\left[(\frac{\mu}{\alpha\beta} + (\frac{\beta-\phi}{\alpha\beta}+1)\frac{\mu}{\gamma} - 1) \pm \sqrt{(\frac{\mu}{\alpha\beta} + (\frac{\beta-\phi}{\alpha\beta}+1)\frac{\mu}{\gamma} - 1)^2 + 4\frac{\mu}{\gamma}(\frac{\beta-\phi}{\alpha\beta}(1-\frac{\mu}{\gamma}) - \frac{\mu+\epsilon}{\alpha\beta})}\right]$

Conditions (a) and (b) derive from the restrictions $B < 0$, $C > 0$ and $B^2 - 4AC \ge 0$ in order to avoid complex values within the square root of the quadratic expression while solving for $m^*_\pm$, the m^* component of the endemic equilibria (i.e., the proportion of members of the total voting population when all three classes coexist). Here, as when solving earlier for $m_\pm$, $A = 1$, $B = \frac{\mu}{\alpha\beta} + (\frac{\beta-\phi}{\alpha\beta}+1)\frac{\mu}{\gamma} - 1$, $C = -\frac{\mu}{\gamma}[\frac{\beta-\phi}{\alpha\beta}(1-\frac{\mu}{\gamma}) - \frac{\mu+\epsilon}{\alpha\beta}]$.

3.5 Threshold Parameters, R_1 and R_2

Our system contains two local thresholds or tipping points where population outcomes, measured as S, V, and M, depend on parameter values. By tipping point we refer to the sociological term that describes the point at which a stable phenomenon turns into a crisis, which in a political context, corresponds to the extreme states of the party: death and growth [11]. In the context of our model, for example, the party can very well die out up until parameter conditions reach $R_1 = 1$ after which point the third party voting and member classes gain individuals. We distinguish between the aforementioned thresholds, R_1 and R_2, by analyzing them qualitatively in a political context.

3.5.1 Interpreting R_1 and R_2

1. $R_1 = \frac{\beta-\phi}{\mu+\epsilon}$: We interpret this threshold value as the net peer pressure, $\beta - \phi$, on susceptibles by individuals of V multiplied by the average time, $\frac{1}{\mu+\epsilon}$, spent in the voting class V. The numerator couples those factors that bear a direct influence (i.e., personal contacts) on the transition between S and V, whereas indirect factors (i.e., secondary contacts such as opposition media and natural exits from the system), μ and ϵ, comprise the denominator. R_1 denotes the average number of susceptibles an individual in V or M would convert if dropped in a homogeneous population of susceptibles.

2. $R_2 = (\frac{\gamma}{\mu})(1 - \frac{1}{R_1})$: We interpret this threshold value as the product of the average time in the voting system, $\frac{1}{\mu}$, the rate of recruiting voters from V into M via influence from party members, γ, and the proportion of the population N in V, $(1 - \frac{1}{R_1})$. Similar to R_1, R_2 measures the average number of V to M conversions per individual in the M class; hence, R_2 is essentially a measure of how effective party members are in recruiting third party voters to become members once there are enough individuals in V.

3.5.2 An Analysis of the Various Conditions of R_1 and R_2

(i) When $R_1 < 1$ the member-free equilibrium, E_2, does not exist since $V < 0$, an unrealistic population. Rather, under such conditions, E_1, the PFE, not only exists but is locally asymptotically stable and the party dies out. However, under certain initial conditions (i.e., a substantial number of M individuals) we note that the party can survive even when $R_1 < 1$ since, for $R_2 = \frac{\gamma}{\mu}(1 - \frac{1}{R_1})$, $R_1 < 1$ also implies $R_2 < 0$. (See related explanation (iii)).

(ii) When $R_1 > 1$ each individual in V and M is converting more than one person in S into V, thus allowing the V class to thrive. In other words, while $R_1 > 1$ renders the PFE unstable it also implies the existence and stability of the MFE (given that $R_2 < 1$) which, as described earlier, does not exist in the political world since a party cannot exist without members. However, as with the case when $R_1 < 1$, we can ensure the coexistence of the S, V and M classes when $R_1 > 1$ under sufficient initial conditions.

(iii) $R_2 < 0$ occurs when $R_1 < 1$ due to the special relationship between R_2 and R_1 where $R_2 = \frac{\gamma}{\mu}(1 - \frac{1}{R_1})$; such a relationship denotes the existence of a negative voting population in the MFE (i.e., no real population in the V class). Since R_2 signifies the conversion of V to M, $R_2 < 0$ implies that M is converting a negative or non-existent V population into the M class, which initially does not make sense in a political context. $R_2 < 0$ seems to correspond to the stability of either the PFE or to the existence of a state where we have individuals M and S and a non-existent or negative V class. However, $R_2 < 0$ can still lead to the stable endemic equilibrium E_3 via a backward bifurcation if there exists a sufficient initial number of individuals in M.

(iv) The condition $R_2 < 1$ guarantees local asymptotic stability of the member-free equilibrium, MFE. For this reason, the condition is not typically conducive to party growth but, as we mentioned before, $R_2 < 1$ may imply the coexistence and stability of E_3. This suggests that R_2 is the more important of the threshold transitions in political terms since it is primarily concerned with the V to M transition where M individuals are more influential in third party voter recruitment by the factor α.

(v) The condition $R_2 > 1$ explains the case where the V and M classes grow by recruiting members from the S and V populations respectively. In other words this condition guarantees that the larger endemic equilibrium exists without the additional conditions (a) and (b) imposed by the quadratic expression of $m^*_\pm$. In a biological analog, this condition describes how M successfully invades both the S and V classes.

3.6 Simulations

After discussing the equilibria for the homogenous one-track model we simulate its various outcomes. We change β to show the various equilibria while fixing the rest of the parameters and initial conditions as follows: $S_0 = 3600$, $V_0 = 1250$, $M_0 = 150$, $\alpha = 1.25$, $\gamma = 0.2$, $\epsilon = 0.25$, $\mu = 0.05$ and $\phi = 0.15$. Note that we employ an ideal set of parameters that we retain for the bifurcation plots in section 3.7.

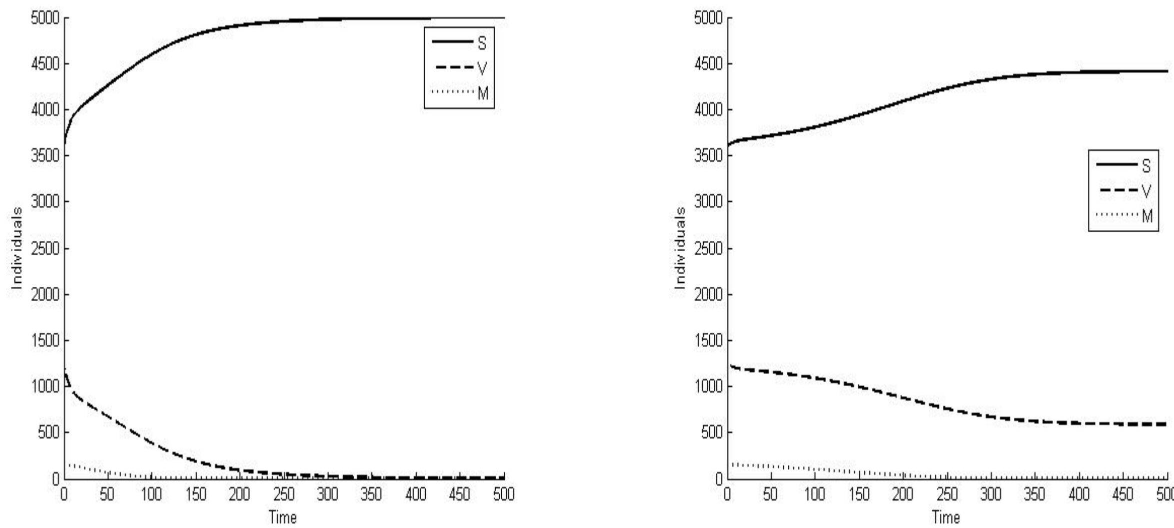

Figure 3: (a) $\beta = 0.44$. (b) $\beta = 0.49$.

Figure 3(a) shows the system approaching the party-free equilibrium (E_1) for $\beta = 0.44$. For this scenario, all individuals eventually return to S and the third party has neither voters nor members.

Figure 3(b) shows the system approaching the member-free equilibrium (E_2) for $\beta = 0.49$. In this case, there are no third party members and all individuals end up in either S or in V.

We can observe the system approaching the stable endemic equilibrium, E_3, in Figure 4(a) where $\beta = 0.55$. Note the coexistence of individuals in each of the three classes, S, V and M. Figure 4(b) shows the system approaching the unstable endemic equilibrium (E_4) for $\beta = 0.50$. Again, individuals exist in all classes; however, due to its unstable nature, the numbers of the individuals in the three classes will stay steady at this equilibrium only if the initial conditions, S_0, V_0 and M_0, equal the values of this equilibrium. For any other initial conditions, the population will never approach this equilibrium (i.e, individuals will only exist at the unstable equilibrium E_4 if they *always* exist at it, a condition that assumes no net changes in the compartment populations).

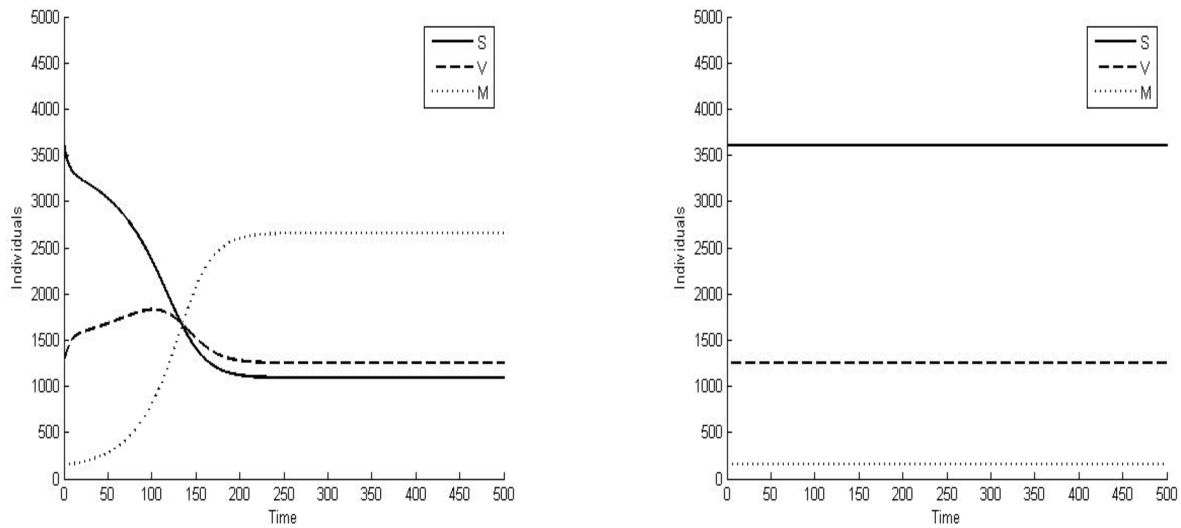

Figure 4: (a) $\beta = 0.55$. (b) $\beta = 0.5$.

Parameter Values					
β	ϕ	μ	α	ϵ	γ
variable	0.15	0.05	1.25	0.25	0.20

Table 4: Parameter set for bifurcation of one-track model.

3.7 Backward Bifurcation

In order to analyze the behavior of our system under varying threshold conditions, we plot M vs. R_2 in Figure 5. To create this plot, we fix all parameter values and vary β as outlined in Table 4.

Figure 5 shows backward bifurcation beginning with $R_2 = 1$. It is important to note the following feature: whenever $R_1 < 1$, $R_2 < 0$, a condition that would normally lead to the death of the party given local asymptotic stability of the PFE, in this case shows the existence of two endemic equilibria, one of which is stable (E_3). In other words, the party can thrive in conditions under which it would normally die out, given that we have the necessary parameters and sufficient initial number of M individuals. Additionally, we qualify our definition of R_2 as the number of third party voter-to-member conversions each M individual realizes. A priori, $R_2 < 0$ seems like a contradiction since we cannot exactly interpret a negative number of conversions,

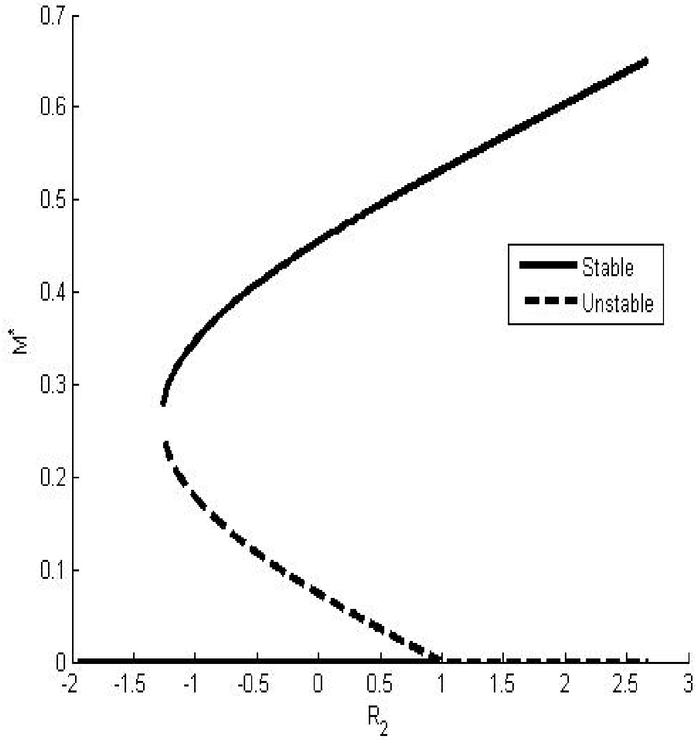

Figure 5: Bifurcation diagram using Table 4 parameter values.

but, for these conditions, V has zero or negative individuals and our traditional definition of R_2 does not apply (see earlier discussion on threshold parameters for more details). Therefore, in political terms, the great surprise to learn is that the party can survive and approach a stable steady state under conditions that would otherwise lead to the death of the party ($R_1 < 1$).

Another interesting scenario is when $R_1 > 1$ but $R_2 < 1$. This situation corresponds to the existence and stability of the MFE, a situation that does not make political sense. However, similar to $R_2 < 0$, these conditions can lead to the coexistence and stability of E_3, given sufficient initial M.

Of course, when $R_1 > 1$ and $R_2 > 1$, we have a stable endemic equilibrium at E_3 and we obtain the ideal conditions needed for the party to thrive.

3.8 Sensitivity Analysis

We apply sensitivity analysis in order to determine the parameters to which our model is most sensitive, in effect allowing us to analyze the effect of the peer-pressure driven recruitment rates of M and V individuals on susceptible transitions to third party voting (i.e., S-to-V transitions) and third party voter transitions to party membership (i.e., V-to-M transitions). This requires that we devise a set of ideal parameter values that produces the desired backward bifurcation for which there exists a stable endemic equilibrium describing the coexistence of the susceptible, voting, and member classes.

We define the sensitivity index of J for a given parameter p as

$$S_p = \frac{\partial J}{\partial p}\frac{p}{J}$$

where p is the parameter in question and J denotes a differentiable functional that depends on the parameter p [1]. In this study we assign the two threshold parameters $R_1 = \frac{\beta-\phi}{\mu+\epsilon}$ and $R_2 = \frac{\gamma}{\mu}(1-\frac{1}{R_1})$ as our functionals for which we want to test parameter sensitivity. We find the sensitivity index via the product of the the partial derivatives of the system's threshold values with respect to each of the parameters and the proportion $\frac{p}{J}$. In this particular case we are not concerned with α since it appears in neither functional, nor with τ since τ emerges only at the two-track level.

Table 5 contains the analytic representations of the sensitivity indices of the parameters with respect to the thresholds R_1 and R_2.

Sensitivity Indices Expressed Analytically		
Parameter	S_p for $R_1 = \frac{\beta-\phi}{\mu+\epsilon}$	S_p for $R_2 = (\frac{\gamma}{\mu})(1-\frac{1}{R_1})$
β	$\frac{\beta}{\beta-\phi}$	$\frac{\beta}{(R_1-1)(R_1)}$
ϕ	$-\frac{\phi}{\beta-\phi}$	$-\frac{\phi}{(R_1-1)(R_1)}$
μ	$-\frac{\mu}{\mu+\epsilon}$	$-\frac{R_1-\frac{\epsilon}{\mu+\epsilon}}{R_1-1}$
ϵ	$-\frac{\epsilon}{\mu+\epsilon}$	$\frac{\epsilon}{(R_1-1)(\mu+\epsilon)}$
γ	0	1

Table 5: Sensitivity indices for R_1 and R_2 of the one-track model.

We apply the set of parameters from Table 4 and vary β in order to produce the sensitivity indices for three conditions of the threshold R_2 (we consider conditions for R_2 since it deals with V-to-M conversion and new individuals in M play a large role in the acquisition of new V individuals during the spread of the party). (See appendix B for sensitivity index values).

We conclude that, regardless of which of the conditions $R_2 < 0$, $0 < R_2 < 1$, or $R_2 > 1$ we deal with, the most sensitive parameter is β, followed by ϕ and γ. Recall that β is the peer driven recruitment rate of susceptibles into the third party voting class, V, by individuals in V while $\alpha\beta$ is the rate at which individuals in S transition into V by persuasion from members. Given the high sensitivity of β we then assume that these recruitment processes dominate the system, and, as a political recommendation, we advise third parties to focus on the degree of these recruitment rates for bringing about third party voting and membership.

Initially, when the party is small, members should first focus on recruiting S into the V class in order to acquire V individuals that, in turn, will become members; these members then recruit susceptibles at an increased factor of $\alpha\beta$. Given this augmented member efficacy in converting individuals in S to V, our advice is naturally to focus the party's endeavors on member recruitment since an increase in the M class will result in a heightened increase in both V and M. In this artificial parameter system ϕ is also very sensitive; however, it is also more difficult to control since it signifies the effects of contacts between third party voters and S individuals in regressing from the V to S classes. We also observe that ϵ is difficult to control since it deals with secondary influences (i.e., the media) on V-to-S regression and third parties can intervene little in opposition advertisements by well funded majority parties. Therefore, to conclude, we advise emphasizing S-to-V transitions until a substantial V population forms at which point the party should focus on γ (i.e., increasing member recruitment that, in turn, bears an even larger impact on voter recruitment).

4 The One-Track Model: A Graphic Summary

We construct Figure 6 by considering the curves specific to $R_1 = 1$, $R_2 = 1$, $B = 0$, and $B^2 - 4AC = 0$ (refer to section 3.4.4 for definitions of A, B, and C). We define q as $q = \frac{\beta-\phi}{\alpha\beta}$, a substitution that facilitates interpretation. We remind the reader that (a) $B < 0$, (b) $B^2 - 4AC > 0$, and $R_2 < 1$ are necessary conditions for the existence of both endemic equilibria E_3 and E_4. Consider the follow regions where we observe equilibria existence and stability:

Refer to section E in the appendix for explanation of figure 6

I In region I where $R_1 < 1$, $R_2 < 1$, and conditions (a) and (b) are satisfied, E_1 and E_3 are stable; depending on the initial conditions the solution tends to one state or the other.

II In region II where $R_1 > 1$, $R_2 < 1$ and conditions (a) and (b) are met, E_2 and

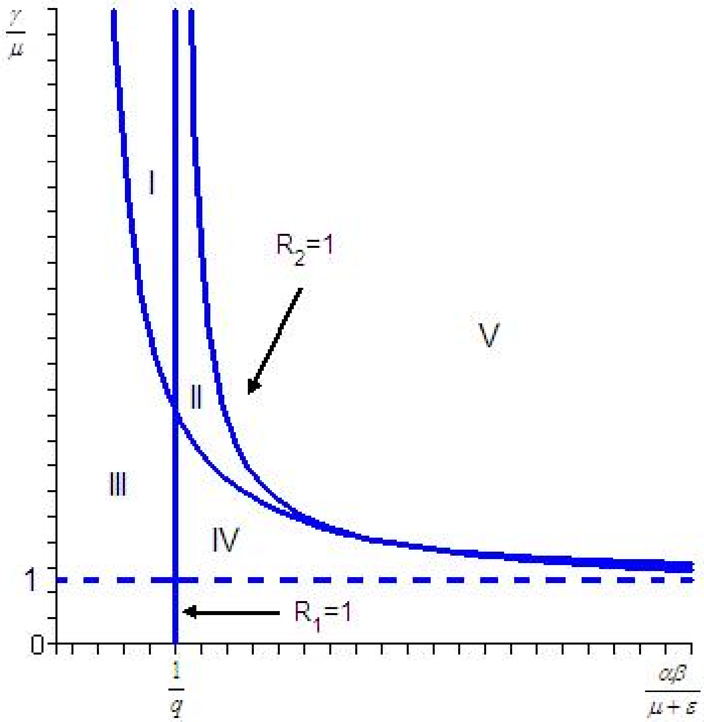

Figure 6: Regions of Equilibria Stability

	E_1	E_2	E_3	E_4
I	Stable	does not exist	Stable	Untable
II	Unstable	Stable	Stable	Unstable
III	Stable	Unstable	does not exist	does not exist
IV	Unstable	Stable	does not exist	does not exist
V	Unstable	Unstable	Stable	does not exist

Table 6: Regions of Equilibria Stability

E_3 coexist as stable equilibria but, if placed in a political context, E_2 is not a realistic outcome for the party.

III In region III E_1 is the only stable equilibrium; the party will always approach the party-free equilibrium.

IV In region IV where $R_1 > 1$, $R_2 < 1$, and conditions (a) and (b) are not met,

E_2, the member-free state, is the only stable equilibrium.

V In region V where both $R_1 > 1$ and $R_2 > 1$ and conditions (a) and (b) are not met, E_3 is the only stable equilibrium; the party will inevitably approach an endemic state.

5 The Green Party of Pennsylvania: A Case Study

After discussing hypothetical parameters in backward bifurcation and sensitivity analysis, we employ real world data from the Green Party to derive parameters and further analyze the model.

5.1 Parameter Estimation for the One-Track Model

In our discussion, we estimate the model's parameters based on the data from the Green Party of Pennsylvania. We use voter registration for the Green Party in place of party membership to distinguish between V and M because we could not obtain data concerning Green Party membership. We initially restricted our complex model to party membership in order to incorporate the 22 states in which voters cannot register to specific parties; however, in the case where access to membership data is limited we measure party registration. In other words, we replace membership with party registration in this particular case study and, for purposes of consistency, we refer to registered voters as M individuals.

After contacting the national Green Party directly, we obtained (1) the number of registered voters in the Green Party of Pennsylvania sampled over the years 2001 to 2005 and (2) access to the number of votes received by all Green Party candidates running in Pennsylvania since the state party's founding. However, due to the lack of data on the *net votes* cast for Green Party candidates we referenced a particular campaign to obtain parameters dependent on the number of individuals in V. Due to limited data we also roughly approximated certain parameters (i.e. the augmentation factor α).

We justify our motivation in choosing the state of Pennsylvania because (1) the total population of Pennsylvania has increased only 1.01 % from 2000 to 2004, a low enough increase that allows us to assume constant population, N, and consistency of social structure (i.e., no resulting movement between H and L) [20] and (2) the data offered from Pennsylvania was free of drastic changes in M that might be associated with exceptional candidates or highly specific events. Our parameter estimations follow:

(1) $\mu = 0.014$: μ, the exit rate from the system, is essentially the average death rate of people in the voting age population. We calculate the parameter by dividing 128,010, the number of deaths in the voting age population in 2002(ages 18 and above since individuals become eligible to vote at 18), by 9,358,833, the number of individuals in the voting age population in 2002 [20]. We apply a unit of $time^{-1}$ to μ.

(2) $\gamma = 115.16$: We assign γ, the recruitment rate of V individuals into the registered class, M, a numerical value by using the relationship $\frac{\Delta M}{\Delta t} = \gamma \frac{VM}{N}$ where ΔM is the increase in M per unit time, $\frac{M}{N}$ is the the proportion of M individuals in the N voting population and V is the total number of voters excluding registered voters. We use the following data to calculate γ : (1)$\frac{\Delta M}{\Delta t}$ is estimated by the increase in Green Party registration over a specific time period (i.e., one year), (2) $\frac{M}{N}$ is derived by dividing the number of Green Party registered individuals by the total voting population N and (3) the total number of Green Party voters, V, is measured by the votes received in a particular election subtracted by the number of registered Green Party voters in the same year (we find the difference in order to isolate simply those who vote Green but are not registered for the party).

Even though we received data of the number of registered Green Party voters in the state of Pennsylvania from 2001 to 2005, we did not have access to the total number of Green Party voters *excluding* registered voters. We avoided this obstacle by referencing the gubernatorial campaign of Michael Morrill on 11/05/2002 in Pennsylvania in which he received 38,030 votes (i.e., 1.1 % of the total votes) and, after subtracting the 3266 registered green party voters at the time [13], we arrive at a voting population V of 34,814 individuals. Note that we assume that all registered Green Party voters voted for Morrill, a relatively safe assumption given the higher level of commitment by registered voters to the party and the relative importance of a gubernatorial election.

In order to determine the proportion $\frac{M}{N}$ we calculated N, the number of voters in the voting age population in 2002, by multiplying 9,358,833, the total number of individuals in the voting age population (ages 18 and up), by the voter turnout for this particular election, 37.39 % (close to the average voter turnout of 38.4 % [21]). We combine these elements in the following quotient: $\gamma = (\frac{\Delta M}{\Delta T})(\frac{V}{\frac{M}{N}})$. Substituting in the appropriate values, $\gamma = \frac{3742}{(0.00093)(34814)}$. After performing the calculation we find that $\gamma = 115.16$, an exceptionally high value implying that the third party will soon become a majority party via such a high initial recruitment rate of members, a rather improbable situation. In other words, this value of γ bodes *too* well for the third party. However, the prediction of an outcome as unlikely as the rapid increase of the third party to the ranks of a majority party provides support for the necessity of the two-track model, which assumes heterogeneous mixing of susceptibles with different affinities to the third party ideology. Therefore, while γ may be initially high due to the availability and high susceptibility of individuals in H, L susceptibles will be particularly resistant to the ideology, in effect, reducing the recruitment into V and thus into M. Over time, a depletion in the H class due to the conversion of its individuals into the V class along with the resistance from the L class will cause γ to decrease with time as the initial fervor of member and voter recruitment decreases. We apply a unit of $time^{-1}$ to γ.

(3) α: We referenced literature regarding the methods and hourly commitment of party members to voter recruitment and approximated that registered voters are roughly three times as effective in recruiting Green Party voters as individuals from V [10]. We do not apply units to α given that α is a scalar.

(4) τ: We do not consider τ for the one-track model since we assume homogeneity of the susceptible population (i.e., we do not have two classes with different recruitment rates from V to S). We do not apply units to τ given that τ is a scalar.

(5) ϵ: We regard ϵ as the linear term representing the role of secondary sources (i.e., the media) in affecting a voter's decision. In the case of the Democratic vote, studies show that during a period of low political information, the predicted probability of a Democratic vote drops from about 0.65 to about 0.55 while at high levels of information (i.e., just prior to elections), there is an independent, media-influenced movement from a 0.4 to 0.6 probability of voting Democrat in 1994 [15]. Assuming that such a phenomenon can be observed in any individual who votes and assuming a medium information flow, we average the difference of the changes in probability during low and high information flow and obtain 0.05 as the value for ϵ. We apply a unit of $time^{-1}$ to ϵ.

It is very difficult to obtain politically realistic projections of the endemic state if we apply these data-derived parameters of which only μ does not incorporate some degree of approximation. With such a high γ value, the one-track model bifurcation plot projects an immediate rise of the third party to majority status. Therefore, while it is useful to derive parameters from actual data, we avoid unrealistic projections for the one-track model by considering the aforementioned ideal set of parameters used in both the backward bifurcation and sensitivity analysis sections. From these idealized parameters we offer advice as to how third parties can strategize and grow by changing their recruitment efforts to match the ideal set of parameters.

6 The Two-Track Model

After thoroughly examining the one-track model, deriving conclusions and providing recommendations, we perform analysis on the original, heterogenous two track model. We begin by obtaining the party free equilibrium (PFE) and the first tipping point, R_1', which is analogous to R_1 in the one track model. We find that determining any other equilibria or thresholds analytically is too complicated and we cannot extract anything politically relevant from further analysis; therefore, we look at equilibria stability numerically by fixing parameter values in the Jacobian matrices. We then present deterministic simulations to observe the outcomes of the various equilibrium conditions. Finally, we offer bifurcation diagrams for the model and analyze the outcomes.

6.1 Analysis

6.1.1 The Party-Free Equilibrium (PFE) and the Threshold/Tipping Point R_1'

In this section, we determine the PFE and calculate the first threshold of the two-track model. Substituting in zero for V_H, V_L, and M in the system of equations we solve for the party-free equilibrium such that E_1 is $(pN, (1-p)N, 0, 0, 0)$. We then use the next generation operator method [4] to solve for the first threshold where

$$R_1' = \frac{1}{2}\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1} - \frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2} + \sqrt{\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1} + \frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]^2 + 4\frac{p\beta_1\beta_2(1-p)}{(\mu+\epsilon_1+(1-p)\phi_1)(\mu+\epsilon_2+p\phi_2)}}\right]$$

(From this point on we use prime superscripts to denote thresholds for the two-track system). See Appendix D for calculations.

R_1' is the first threshold of the two-track model with tracks deriving from the susceptible classes H and L that do not intersect until the M compartment (i.e, no interaction between individuals of different susceptibilities until subscribing membership to the third party). We refer to R_1 when defining R_1' as the number of susceptibles an individual in V_H or V_L converts into either voting class if dropped in a population of H and/or L. In the expression for R_1' we try to make sense of the recurring terms $\frac{1}{2}\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1}\right]$ and $\left[\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]$. With the exception of a few differences, each term assumes the form of R_1 of the one-track system where

$R_1 = \frac{\beta-\phi}{\mu+\epsilon}$; for that reason, we designate the first recurring term as R'_H (concerned with the dynamics of the H track) and the second term R'_L (concerned with the dynamics of the L track). We treat these tracks separately because R'_1 is primarily concerned with the susceptible transition to the third party voting stage at which the tracks remain independent (i.e., individuals from different tracks do not .

The expressions' numerators are $(\beta_1-\phi_1)p$ and $(\beta_2-\phi_2)(1-p)$, each representing the net influence of direct contacts in converting p susceptibles in H to V_H and converting $(1-p)$ individuals in L to V_L respectively. Additionally, both denominators incorporate the terms $(\mu+\epsilon_1+(1-p)\phi_1)$ and $(\mu+\epsilon_2+p\phi_2)$; these terms describe the net rate at which voters leave the voting class due to secondary influences (i.e, natural death rate, μ, and the media, ϵ). Unlike R_1, the denominators also incorporate an additional ϕ term. In the case of R'_H the additional term $(1-p)\phi_1$ accounts for the rate at which opposition voters influence V_H to regress back into H. We multiply ϕ_1 by the proportion of N, $(1-p)$, since it accounts for the influence of L on third party voter regression. In the case of R'_L, $p\phi_2$ accounts for the rate at which direct contacts with members of the opposition, specifically those from H, drive voters in V_L back into L (hence the factor p since H enters the population at a proportion p of the toting voting population N).

Having explained the main differences between $R_{H/L}$ and R_1 we conclude that both terms independently describe the thresholds for each track H and L respectively *in addition to* special components in terms of ϕ that account for cross-track influences specific to the two-track model. Despite these simplifications we still face a complicated expression in interpreting R'_1 term-by-term; however, we bypass this obstacle by bounding R'_1 and interpreting these bounds which are easier to translate into political terms. Observe how we avoid analysis of the $\sqrt{B^2-4AC}$ term by interpreting R'_1 via the following bounds:

$$\frac{1}{2}\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1}+\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]+\frac{1}{2}|\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1}-\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]|\leq R'_1$$

and

$$R'_1\leq\frac{1}{2}\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1}+\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]+\frac{1}{2}|\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1}-\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]|+\sqrt{\frac{p\beta_1\beta_2(1-p)}{(\mu+\epsilon_1(1-p)\phi_1)(\mu+\epsilon_2+p\phi_2)}}$$

Applying what we now know of R'_H and R'_L we evaluate the terms in the lower bound. Depending on which of the terms is greater (largely dependent on the proportion of the voting population entering each track) one set of terms will cancel, leaving either R'_H or R'_L. In a political context this translates to the dominance of a certain susceptible population over the dynamics of the entire system. We interpret R'_H or R'_L as the lower bound by reasoning that R'_1 must be at least as large as the dominant threshold R'_H or R'_L.

The upper bound considers both (1) the dominant threshold term of the lower bound and (2) an additional expression $\sqrt{\frac{p\beta_1\beta_2(1-p)}{(\mu+\epsilon_1(1-p)\phi_1)(\mu+\epsilon_2+p\phi_2)}}$. Similar to the additional ϕ terms in the denominator of the R'_H and R'_L, (2) is a cross-track term that accounts for interaction between the two tracks. Analytically, it translates to the geometric mean of both thresholds that, when added to the dominant threshold, constitutes the upper bound of R'_1.

In order for the PFE to be unstable we need $R'_1 > 1$. In other words, using the threshold definition from the one-track model, if an individual from either V_H, V_L or M is introduced in a population of H and L, he/she must convert more than one susceptible into third party voters in order to establish the V class. Similar to the one-track model, a sufficient initial number of members can bypass stability of the PFE as can be observed in the backward bifurcation section for the two-track model. (See section 5.3). Due to the parallels between the one and two-track R_1 and R'_1 thresholds most recommendations on voter recruitment made earlier in the sensitivity analysis section of the one-track model pertain to the two-track model.

6.1.2 The Threshold/Tipping Point R'_2

We now find the threshold parameter R'_2. Assuming the same interpretation of R_2 from the one-track model, we define R'_2 as the number of voting individuals a member can convert into M if dropped in a population of V_H and/or V_L. Since R'_2 is primarily concerned with the transition from third party voting to membership we conveniently regard M as the only infectious class, in effect facilitating the next generation operator method in determining this threshold.

The following method solves for threshold parameter R'_2.

$$\frac{dM}{dt} = \gamma\frac{(V_H + V_L)}{N} - \mu M$$

The Jacobian is

$$J = \gamma\left(\frac{V_H^* + V_L^*}{N}\right) - \mu$$

Therefore, we define

$$M = \gamma\left(\frac{V_H^* + V_L^*}{N}\right)$$

and

$$D = \mu$$

$$MD^{-1} = \frac{\gamma}{\mu}\left(\frac{V_H^* + V_L^*}{N}\right)$$

Since the matrix only has one eigenvalue we conclude that

$$R_2' = MD^{-1} = \frac{\gamma}{\mu}\left(\frac{V_H^* + V_L^*}{N}\right)$$

Note the similarity of R_2' to R_2 of the one-track model from which we expected something of the form $\frac{\gamma}{\mu}(1 - \frac{1}{R_1})$. While R_2' is not expressed explicitly, given the use of the terms V_H^* and V_L^*, we can still make conclusions about the two-track model numerically.

6.1.3 Stability of Equilibria Given Fixed Parameters

We require that the Jacobian of the proportionalized two-track model be:

$J =$

$$\begin{pmatrix} \phi_1\tau v_h - \beta_1 I - \mu & \phi_1 v_h & \epsilon_1 + \phi_1(\tau h + l) - \beta_1 h & -\beta_1 h & -\alpha\beta_1 h \\ \phi_2 v_l & \phi_2\tau v_l - \beta_2 I - \mu & -\beta_2 l & \epsilon_2 + \phi_2(\tau l + h) - \beta_2 l & -\alpha\beta_2 l \\ \beta_1 I - \phi_1\tau v_h & -\phi_1 v_h & \beta_1 h - \epsilon_1 - \phi_1(\tau h + l) - \gamma m - \mu & \beta_1 h & \alpha\beta_1 h - \gamma v_h \\ -\phi_2 v_l & \beta_2 I - \phi_2\tau v_l & \beta_2 l & \beta l - \epsilon_2 - \phi_2(\tau l + h) - \gamma m - \mu & \alpha\beta_2 l - \gamma v_l \\ 0 & 0 & \gamma m & \gamma m & \gamma v_h + \gamma v_l \end{pmatrix}$$

where $I = v_h + v_l + \alpha m$.

As in the one-track case, the two-track model produces two endemic equilibria, a member-free equilibrium and a party-free equilibrium for the ideal set of parameters of Table 7. Note the following unit for β_1, β_2, ϕ_1, ϕ_2, μ, ϵ_1, ϵ_2, and γ: $time^{-1}$ (α, τ, p are scalars).

Parameter Values										
p	β_1	β_2	ϕ_1	ϕ_2	μ	α	ϵ_1	ϵ_2	γ	τ
0.4	0.6	0.3	0.14	0.175	0.05	3	0.16	0.24	0.22	1.25

Table 7: Parameter set 1 for two-track model.

We arrive at the following proportionalized equilibrium points:

$$e_i = (\frac{H^*}{N}, \frac{L^*}{N}, \frac{V_H^*}{N}, \frac{V_L^*}{N}, \frac{M^*}{N}),$$
$$e_1 = (0.242, 0.484, 0.144, 0.106, 0.024),$$
$$e_2 = (0.030, 0.100, 0.106, 0.144, 0.620),$$
$$e_3 = (0.381, 0.589, 0.019, 0.011, 0),$$
$$e_4 = (0.4, 0.6, 0, 0, 0).$$

After substituting in e_1, e_2, e_3 and e_4 into the Jacobian we get the eigenvalues as listed in Table 8.

	e_1	e_2	e_3	e_4
λ_1	-0.52	-1.48	-0.05	-0.05
λ_2	-0.21	-0.94	-0.05	-0.05
λ_3	0.016	-0.05	-0.69	0.010
λ_4	-0.05	-0.08	-0.90	-0.45
λ_5	-0.05	-0.15	-0.05	-0.05

Table 8: Eigenvalues of equilibria found with parameter values of Table 7.

In order to have local asymptotic stability for an equilibrium point we require that all eigenvalues have negative real parts. Since λ_3 is positive for e_1 and e_4, then these equilibria points are unstable. The eigenvalues of the other two equilibria, e_2 and e_3, are all negative and thus e_2 and e_3 are LAS. Thus we have two endemic equilibria, e_2 and e_1, one LAS and the other not, one LAS member-free equilibrium and one unstable party-free equilibrium.

Additionally, employing a different set of parameters in Table 9, in order to observe changes in equilibria behavior

we get the following equilibrium points:

$$e_1 = (0.068, 0.770, 0.017, 0.067, 0.078),$$
$$e_2 = (0.008, 0.157, 0.009, 0.074, 0.752),$$

Parameter Values										
p	β_1	β_2	ϕ_1	ϕ_2	μ	α	ϵ_1	ϵ_2	γ	τ
0.1	0.4	0.2	0.15	0.1875	0.05	3	0.3	0.45	0.6	1.2

Table 9: Parameter set 2 for two-track model.

$$e_3 = (0.1, 0.9, 0, 0, 0).$$

Note the following unit for β_1, β_2, ϕ_1, ϕ_2, μ, ϵ_1, ϵ_2, and γ: $time^{-1}$ (α, τ, and p are scalars).

When we substitute e_1, e_2 and e_3 into the Jacobian we get the eigenvalues of Table 10.

	e_1	e_2	e_3
λ_1	-0.70	-1.46	-0.05
λ_2	-0.58	-1.26	-0.05
λ_3	0.034	-0.34	-0.40
λ_4	-0.06	-0.05	-0.60
λ_5	-0.05	-0.15	-0.05

Table 10: Eigenvalues of equilibria found with parameter values of Table 9.

Since λ_3 is positive for e_1, then this equilibrium point is unstable. The eigenvalues of the other two equilibria, e_2 and e_3, are all negative and so are LAS. Thus we have two endemic equilibria, e_2 and e_1, one LAS and the other not, and one LAS party-free equilibrium.

6.2 Simulations

We apply the following initial conditions and parameters to all four simulations: $H_0 = 1210$, $L_0 = 2420$, $V_{H_0} = 720$, $V_{L_0} = 530$, $M_0 = 120$, $p = 0.4$, $\alpha = 3$, $\beta_2 = 0.3$, $\gamma = 0.2$, $\epsilon_1 = 0.16$, $\epsilon_2 = 0.24$, $\mu = 0.05$, $\phi_1 = 0.14$, $\phi_2 = 0.175$ and $\tau = 1.25$. Note the following unit for β_1, β_2, ϕ_1, ϕ_2, μ, ϵ_1, ϵ_2, and γ: $time^{-1}$ (α, τ, and p are scalars and initial conditions have units of *individuals*). We shaped the parameters to suit the condition that $R_1 > 1$ because this condition guarantees existence of E_2 in addition to the other equilibria; therefore, we can plot all four equilibria of the system. In the following simulations we vary β_1 in order to show the various equilibria that we expect to exist as based on the one-track model.

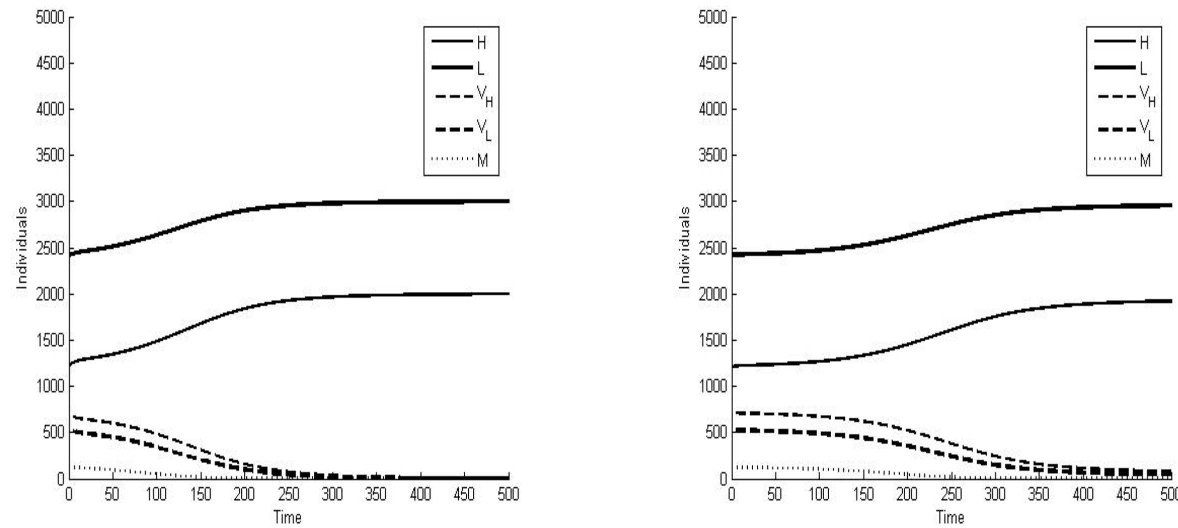

Figure 7: (a) $\beta_1 = 0.55$. (b) $\beta_1 = 0.59$.

Figure 7(a) shows the system approaching a party-free equilibrium (similar to E_1) for $\beta_1 = 0.55$. In this case, all individuals end up in the susceptible classes and the third party has neither voters nor members. Figure 7(b) shows the system approaching a member-free equilibrium (similar to E_2) for $\beta_1 = 0.59$. In this case, there are no third party members and all individuals end up in either the susceptible or third party voting classes.

Figure 8(a) shows the system approaching a stable endemic equilibrium (analogous to E_3 in the one-track model) for $\beta_1 = 0.65$. This state describes a successful coexistence among individuals of all five classes H, L, V_H, V_L, and M. Figure 8(b) shows the system approaching the unstable endemic equilibrium (similar to E_4) for $\beta_1 = 0.60$. Again, in this case there are individuals in all classes. Note, however, that the numbers of the individuals in the five classes will stay steady at this equilibrium only if the initial conditions, H_0, L_0, V_{H_0}, V_{L_0} and M_0, equal the values of this equilibrium. Given any other set of initial conditions this equilibrium will not be reached (i.e., individuals will only exist at the unstable equilibrium E_4 if they always exist at it, a condition that assumes no net changes in the compartment populations).

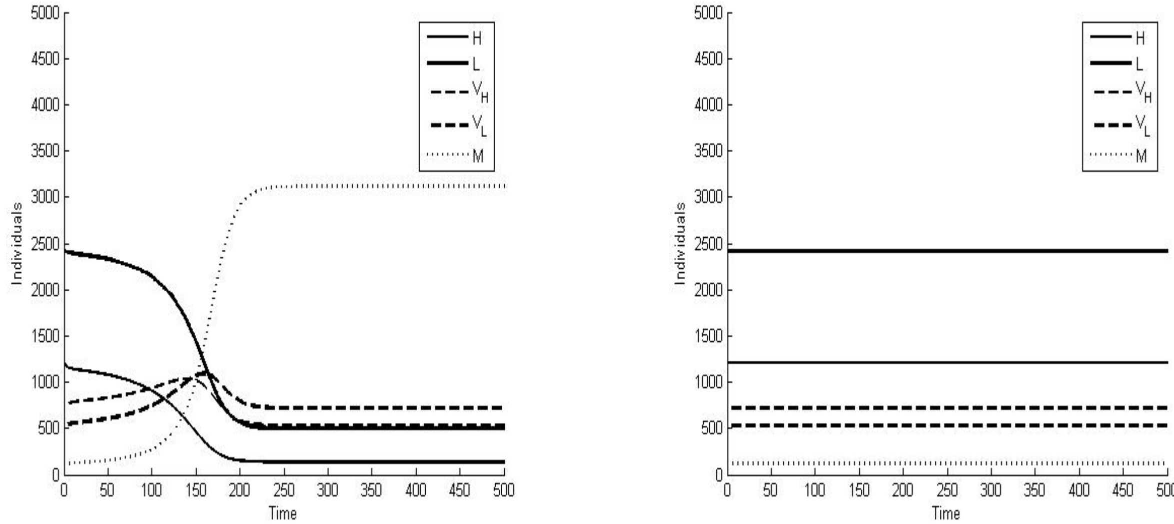

Figure 8: (a) $\beta_1 = 0.65$. (b) $\beta_1 = 0.6$.

It is important to note several considerations when looking at the simulations. First of all, since we were unable to analytically determine all of the equilibria and we did not set out to prove the existence of more equilibria, we cannot be sure that we have considered all possible equilibria. Additionally, we required fractional values for the initial conditions in order to demonstrate the E_4 analogue but, since we cannot realistically have fractions of individuals, we rounded the values to the integers listed above.

6.3 Bifurcation Diagrams

(a) We created the set of parameters in Table 11 in order to maintain backward bifurcation for the two-track system. Note the following unit for β_1, β_2, ϕ_1, ϕ_2, μ, ϵ_1, ϵ_2, and γ: $time^{-1}$ (α, τ, and p are scalars)

Parameter Values										
p	β_1	β_2	ϕ_1	ϕ_2	μ	α	ϵ_1	ϵ_2	γ	τ
0.1	0.4	0.2	0.15	0.1875	0.05	3	0.3	0.45	variable	1.2

Table 11: First set of bifurcation parameter values.

In order to create the above parameter set we varied β, ϵ, and ϕ, approximating the degree by which the corresponding parameters varied. We used intuition in order to approximate the relative factor by which, for example, β_1 exceeds β_2. As it turns out, our approximations of the relative differences between corresponding parameters still maintained backward bifurcation.

We started off by assuming that the portion of the population entering the H class, p, was 0.1. This decision was motivated by (1) the consideration that the Green Party, given its highly progressive, typically youth-catering agenda directly appeals to about 10 percent of the total voting population (i.e, individuals in a young, educated, progressive social sphere) and (2) the parameter value 0.1 gives us a more reasonable endemic equilibrium that does not approach majority status (approximately 1) as quickly as larger values for p. In determining relative β parameters we assumed that H susceptibles are twice as likely to become voters as L susceptibles; this lower relative value is appropriate given that our methodology of placing individuals into H and L via the number of low versus high affinity factors of the individual lessens the magnitude of difference between the susceptible classes. For ϵ we assumed that opposition media and other secondary influences would factor 1.5 times more in V_L regression back to L than in the transition of V_H back to H. Likewise, ϕ_2 is 1.25 times greater than ϕ_1 because, as with ϵ, ϕ is greater for L susceptibles who are more resistant to third party ideology.

Figure 9, the bifurcation diagram for this parameter set, shows the possible behaviors of the member class M by varying γ. We observe that the minimum value of γ that would give hope to the party is 0.28 because neither endemic equilibrium exists for $\gamma < 0.28$. However, given $\gamma > 0.28$ and a certain initial number of members M, the party can approach a stable endemic equilibrium. We observe the general trend that the larger our value of γ the larger the values of the endemic equilibrium. An important conclusion we make concerning the parallels between the one-track and two-track models is how, even when $R_1 < 1$, the party can thrive in both tracks given that γ is large enough.

(b) A second set of parameters in Table 12 considers the case when $p = 0.05$, a condition in which the third party targets a high affinity class that comprises only 5 percent of the total voting population:

We start off by assuming that the portion of the population entering the H class, p, is 0.05, a situation pertaining to third parties with highly specific,

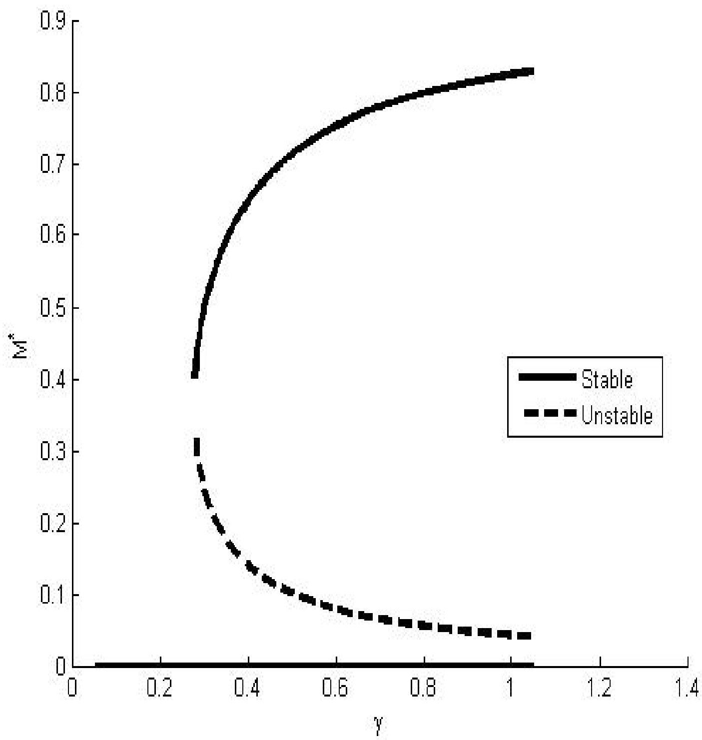

Figure 9: Bifurcation diagram using parameters of Table 11

Parameter Values										
p	β_1	β_2	ϕ_1	ϕ_2	μ	α	ϵ_1	ϵ_2	γ	τ
0.05	0.2	0.09	0.15	0.19	0.05	3	0.5	0.7	variable	1.25

Table 12: Second set of bifurcation parameter values.

somewhat extreme agendas that do not target the majority vote. As with before, we define relative parameter values aiming to maintain backward bifurcation. We establish these relations such that β_1 exceeds β_2 by a factor of 2.2, ϵ_2 affects V_L 1.4 times more than ϵ_1 influences V_H, and ϕ_2 is 1.25 times as large as ϕ_1.

Figure 10 shows the bifurcation diagram for this parameter set. If we apply the above parameters to our expression of R_1 we find that $R_1 = 0.1013$ which,

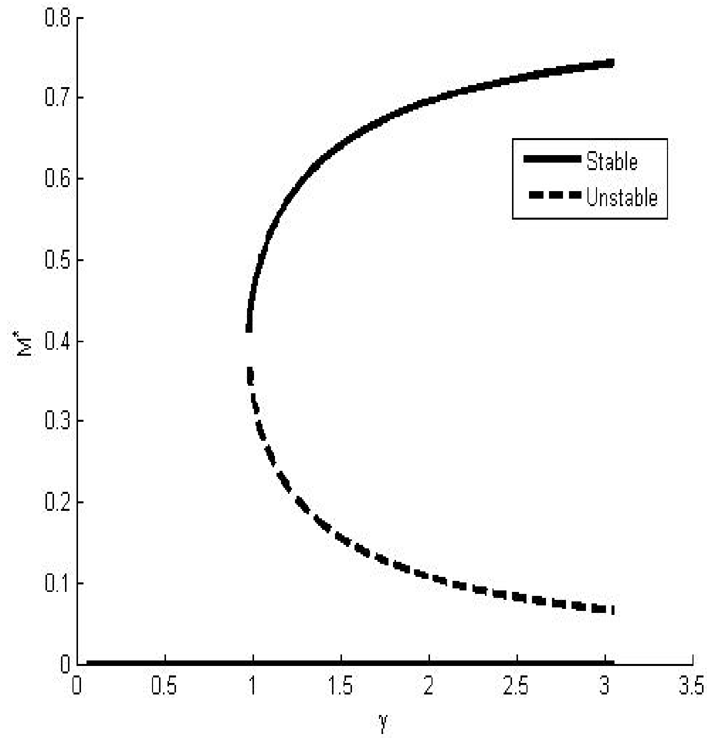

Figure 10: Bifurcation diagram using parameters of Table 12

given that the PFE is locally asymptotically stable for $R_1 < 1$, describes the death of the party. Not only does the initial high-affinity susceptible population comprise 5 percent of the total voting population, the rate of susceptible transition to third party voting embodied in the β parameters is very low. The fact that we produce a bifurcation plot showing endemic equilibria existence, however, shows that even when transition rates from the susceptible to voting classes are low, the party can still survive if γ is large enough.

7 Conclusion

We created a deterministic model in order to represent the effects of member and third party voter recruitment of voters in opposing parties in the spread of a third party. Given its initial complexity we simplified the model to see how analysis of the one-track version sheds insight into the two-track model system. We used data to estimate parameters but, due to the lack of available information concerning net votes cast for Green Party candidates, our estimated parameters provided little valuable information. Therefore, instead we created ideal sets of parameters for both the one and two-track systems in order to obtain a state of class coexistence and translated these ideal parameters into political terms via strategies that parties can take to initiate growth. Many of our recommendations derive from considering several scenarios the party might take in terms of the thresholds or tipping points R_1, R_2, and R_1' (analogous to R_1 of the simple model). We regard these tipping points as decisive factors in the behavior of the party since they explain how voter recruitment and opposition efforts, embodied in the values of the system's parameters, can drive a voting population to death, growth, or unstable stagnancy (i.e., the steady state E_3) depending on the parameters used. Hence, we translate parameters of relative magnitude into strategies that politicians can use in spreading a third party.

For example, consider a voting population N. Mathematically speaking, the population can assume four different outcomes: the party dies out, members die out but opposition voters and party voters remain, and all classes coexist in either an unstable or stable state. In the political context, however, we only consider two of the possible outcomes since (1) a member-free equilibrium implies that voters vote for a party that does not exist (where we assume that parties require members for existence) and (2) the population cannot tend to and stay at the unstable endemic equilibrium unless the number of individuals in each compartment stays fixed (a highly unlikely scenario given the local dynamics of the system where individuals travel between classes).

We now have two possible outcomes: party death or party growth. For the first option, the party dies out if $R_1 < 1$ and the population will always tend to this state given that the net flow into V, measured as $\beta - \phi$ is less than opposition from the media and the natural exit rate of voters from the system embodied in $\mu + \epsilon$. Therefore, the party dies out *unless* a stable endemic equilibrium also exists that guarantees a stable state of class coexistence at which susceptibles, voters, and members exist (i.e., the party grows because it has voters and members needed to recruit susceptibles). Backward bifurcation plots of both the one and two-track systems demonstrate the coexistence of the PFE and endemic equilibrium. Therefore, the party can assume two tracks: death or growth where the deciding factor is the initial number of members in the system that, if large enough, can overcome the party's tendency to death.

The case where $R_1 > 1$ and $R_2 < 1$ is similar to the previous case except that there is an additional stable member-free equilibrium. However, as we stated earlier, we will not be considering this scenario in a political context.

On the other hand, if $R_2 > 1$, then only the endemic equilibrium will be stable. Then, so long as there is at least one member, the result will be coexistence of all classes. However, this does not seem likely since all bifurcation diagrams that show this case would imply that a majority of the voting population would become third party members. Given that we study a third party, this outcome seems unrealistic.

In addition to running deterministic simulations of both systems that show the tendency of the population to all four equilibria states, we performed sensitivity analysis to determine those parameters to which our system was most sensitive. Not surprisingly, β, the recruitment rate of susceptibles into the voting class by voters and members (with the augmentation factor α), emerged as the most sensitive parameter, followed by ϕ and γ. This indicates that the S-to-V transition is most important initially since it builds up the population of V. Only after V has accumulated a substantial number of individuals should the party focus on the V to M transition by accruing members with an effort γ. Then, once more members have joined, the party grows even faster since members recruit opposition voters at an increased factor α.

Even though we used available data to estimate parameters, we ultimately used ideal parameters to model the desired state of endemic coexistence among all classes. Our model does not aim to predict the future of third parties, but rather to offer recruitment strategies to parties so that they might grow and spread within a voting population. We conclude with the final advice that parties should build up the number of initial members in the party since, if $R_1 < 1$, this value ultimately determines the fate of the party: death or growth.

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A Equilibria

A.1 Proving the Existence of Two Endemic Equilibria

(a) First we verify that $f(1) > 0$ where

$$f(m^*) = m^{*2} + \left[\left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1-\frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta}\right]m^* - \frac{\mu}{\gamma}\left[\frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right) - \frac{\mu+\epsilon}{\alpha\beta}\right] = 0$$

$$f(1) = 1 + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - \left(1-\frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta} - \frac{\mu}{\gamma}\frac{\beta-\phi}{\gamma\beta}\left(1-\frac{\mu}{\gamma}\right) + \frac{\mu}{\gamma}\frac{\mu+\epsilon}{\alpha\beta}$$

Note that, earlier we required $\mu < \gamma$ and $\beta > \phi$; therefore $\frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - \frac{\mu}{\gamma}\frac{\beta-\phi}{\gamma\beta}(1-\frac{\mu}{\gamma}) > 0$ holds true since $(1-\frac{\mu}{\gamma}) < 1$ and the expression essentially subtracts from the first term a fraction of itself.

Therefore,

$$f(1) = 1 - \left(1-\frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta} + \frac{\mu}{\gamma}\frac{\mu+\epsilon}{\alpha\beta} + P$$

where we define $P = \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - \frac{\mu}{\gamma}\frac{\beta-\phi}{\gamma\beta}(1-\frac{\mu}{\gamma}) > 0$. After simplifying,

$$f(1) = \frac{\mu}{\gamma} + \frac{\mu}{\alpha\beta} + \frac{\mu}{\gamma}\frac{\mu+\epsilon}{\alpha\beta} + P$$

which is the sum of positive parameters, thus verifying that $f(1) > 0$.

(b) Now we aim to show that $f'(1) > 0$ where $f'(m^*) = 2m^* + (\frac{\beta-\phi}{\alpha\beta})\frac{\mu}{\gamma} - (1-\frac{\mu}{\gamma}) + \frac{\mu}{\alpha\beta}$:

$$\begin{aligned} f'(1) &= 2 + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - 1 + \frac{\mu}{\gamma} + \frac{\mu}{\alpha\beta} \\ &= 1 + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} + \frac{\mu}{\gamma} + \frac{\mu}{\alpha\beta} \end{aligned}$$

which is a sum of positive terms since $\beta > \phi$. Therefore, $f'(1) > 0$.

Since $f'(1) > 0$ and $f(1) > 0$, then the roots of $f(m^*)$ are less than 1.

(c) Using the quadratic formula we get

$m^*_\pm = \frac{1}{2}(-B \pm \sqrt{B^2 - 4AC})$

where

$$A = 1$$

$$B = \left(\frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1 - \frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta}$$

$$C = -\frac{\mu}{\gamma}\left[\frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right) - \frac{\mu + \epsilon}{\alpha\beta}\right]$$

Assuming $B < 0$ then the vertex, $\frac{-B}{2}$, of $f(m^*)$ is positive and, assuming that $B^2 - 4AC > 0$ so that $f(m^*)$ has two real solutions, we infer that at least one solution of $f(m^*)$ is positive. We continue noting that since $C > 0$ then $f(0) > 0$ and therefore both solutions are positive. Given two positive solutions that are less than 1 we then claim that both solutions are in $(0, 1)$; hence the existence of two positive endemic equilibria under the conditions: $B < 0$, $C > 0$ and $B^2 - 4AC > 0$. Now that we have shown that $B < 0$, $C > 0$ and $B^2 - 4AC > 0$ support the existence of two positive solutions, we need to find under what conditions those statements are true.

(d) $B < 0$, $C > 0$ and $B^2 - 4AC > 0$ will be true under the following conditions:

$$C > 0$$

if and only if

$$-\frac{\mu}{\gamma}\left[\frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right) - \frac{\mu + \epsilon}{\alpha\beta}\right] > 0$$

$$(\beta - \phi)\left(1 - \frac{\mu}{\gamma}\right) < (\mu + \epsilon)$$

$$\frac{\beta - \phi}{\mu + \epsilon}\left(1 - \frac{\mu}{\gamma}\right) < 1$$

$$1 - \frac{\mu}{\gamma} < \frac{\mu + \epsilon}{\beta - \phi}$$

$$1 - \frac{\mu + \epsilon}{\beta - \phi} < \frac{\mu}{\gamma}$$

$$R_2 = \frac{\gamma}{\mu}\left(1 - \frac{\mu + \epsilon}{\beta - \phi}\right) < 1$$

We then impose $B < 0$ as an additional condition for the existence of endemic equilibria.

$$B < 0$$

if and only if

$$\frac{1}{\alpha\beta} + \left(\frac{\beta-\phi}{\alpha\beta} + 1\right)\frac{1}{\gamma} < \frac{1}{\mu}$$

which, again, translates to $B < 0$ where B is defined on the previous page.

We require that $B^2 - 4AC \geq 0$ for the existence of m^* (in order to avoid complex values of $m^*_{\pm}$). Substituting in the values of A, B and C we get:

$$\left[\frac{\mu}{\alpha\beta} + \left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - 1\right]^2 - 4\frac{\mu}{\gamma}\left[\frac{\mu+\epsilon}{\alpha\beta} - \frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right)\right] \geq 0$$

Expansion and summing like terms yields

$$\left(\frac{\mu}{\alpha\beta}\right)^2 + \left(\frac{\mu}{\gamma}\right)^2 - 2\left(\frac{\mu}{\gamma}\right)^2\frac{\beta-\phi}{\alpha\beta} + \left(\frac{\mu}{\gamma}\right)^2\left(\frac{\beta-\phi}{\alpha\beta}\right)^2 + 1 -$$

$$2\frac{\mu}{\alpha\beta}\frac{\mu}{\gamma} + 2\frac{\mu}{\alpha\beta}\frac{\mu}{\gamma}\frac{\beta-\phi}{\alpha\beta} - 2\frac{\mu}{\alpha\beta} - 2\frac{\mu}{\gamma} + 2\frac{\mu}{\gamma}\frac{\beta-\phi}{\alpha\beta} - 4\frac{\mu}{\gamma}\frac{\epsilon}{\alpha\beta} \geq 0$$

Then by completing the square we get the following equations

$$\left[\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1\right]^2 \geq 4\frac{\mu}{\alpha\beta}\left(1+\frac{\epsilon}{\gamma}\right)$$

Next we take the square root of both sides,

$$\left|\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1\right| \geq 2\sqrt{\frac{\mu}{\alpha\beta}\left(1+\frac{\epsilon}{\gamma}\right)}$$

Since we require $\mu < \gamma$ and $\beta > \phi$, then

$$\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1 = \frac{\mu}{\alpha\beta} + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} + \left(1-\frac{\mu}{\gamma}\right) > 0$$

and so

$$\left|\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1\right| = \frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1$$

Then replacing the absolute value terms and rearranging we get

$$\frac{\mu}{\gamma} \leq \frac{\mu}{\alpha\beta} + \frac{\beta-\phi}{\alpha\beta}\frac{\mu}{\gamma} - 2\sqrt{\frac{\mu}{\alpha\beta}\left(1+\frac{\epsilon}{\gamma}\right)} + 1$$

A.2 Determining the Stability of E_3 and E_4

Let

$$A' = 1$$

$$B' = \left[-(\beta-\phi)\left(1-\frac{2\mu}{\gamma}-m^*\right)+\alpha\beta m+\mu+\epsilon+\gamma m^*\right]$$

$$C' = -\gamma m^*\left[\alpha\beta\left(1-\frac{\mu}{\gamma}-2m^*\right)-\frac{(\beta-\phi)\mu}{\gamma}-\mu\right]$$

The solutions of the characteristic equation are the eigenvalues of $J(\frac{\mu}{\gamma}, m^*_\pm)$, and the stability of the endemic equilibria depend of the signs of their real parts. If the real parts of both eigenvalues are negative then the we conclude that the endemic equilibrium is locally asymptotically stable.

The eigenvalues are given by

$$\lambda_\pm = \frac{1}{2}(-B' \pm \sqrt{B'^2-4C'})$$

We note that if $B'^2-4C' < 0$, then the endemic equilibrium is locally asymptotically stable if $B' > 0$. On the other hand, if $B'^2-4C' \geq 0$ then the endemic equilibrium is locally asymptotically stable if $B' > 0$ and $C' > 0$.

We now prove that $B' > 0$ always holds given $m^*_\pm > 0$ (i.e., given $m^*_\pm$ exists).

$B' > 0$ holds when

$$\left[-(\beta-\phi)\left(1-\frac{2\mu}{\gamma}-m^*_\pm\right)+\alpha\beta m^*_\pm+\mu+\epsilon+\gamma m^*_\pm\right] > 0$$

$$m^*_\pm((\beta-\phi)+\alpha\beta+\gamma) > (\beta-\phi)-2\frac{\mu}{\gamma}(\beta-\phi)-(\mu+\epsilon)$$

$$m^*_\pm > \frac{(\beta-\phi)-2\frac{\mu}{\gamma}(\beta-\phi)-(\mu+\epsilon)}{(\beta-\phi)+\alpha\beta+\gamma}$$

$$m^*_\pm > \frac{\frac{\beta-\phi}{\alpha\beta}-2\frac{\mu}{\gamma}\frac{\beta-\phi}{\alpha\beta}-\frac{\mu+\epsilon}{\alpha\beta}}{\frac{\beta-\phi}{\alpha\beta}+1+\frac{\gamma}{\alpha\beta}} = \frac{-\frac{\gamma}{\mu}C-\frac{\mu(\beta-\phi)}{\gamma\alpha\beta}}{\frac{\beta-\phi}{\alpha\beta}+1+\frac{\gamma}{\alpha\beta}}$$

Recall that $C = -\frac{\mu}{\gamma}[\frac{\beta-\phi}{\alpha\beta}(1-\frac{\mu}{\gamma})-\frac{\mu+\epsilon}{\alpha\beta}]$.

Note that $C > 0$ (for the condition that $R_2 < 1$), $\beta > \phi$, and $m^*_\pm > 0$ (refer to the Appendix A.1, where we show that $m^*_\pm$ is contained in (0,1)). Therefore,

$$\frac{-\frac{\gamma}{\mu}C - \frac{\mu(\beta-\phi)}{\gamma\alpha\beta}}{\frac{\beta-\phi}{\alpha\beta} + 1 + \frac{\gamma}{\alpha\beta}} < 0$$

and so $B' > 0$ whenever m^* is greater than a negative expression. Since this always holds, then $B' > 0$ is always true.

We now verify the $C' > 0$ criterion for local asymptotic stability of the endemic equilibria where $R_2 < 1$ whenever $m^* > -\frac{B}{2}$.

Recall that $B = (\frac{\beta-\phi}{\alpha\beta})\frac{\mu}{\gamma} - (1 - \frac{\mu}{\gamma}) + \frac{\mu}{\alpha\beta}$.

$C' > 0$ on condition that

$$-\gamma m^* \left[\alpha\beta\left(1 - \frac{\mu}{\gamma} - 2m\right) - \frac{(\beta-\phi)\mu}{\gamma} - \mu\right] > 0$$

$$-\alpha\beta + \frac{\mu}{\gamma}\gamma\beta + 2m\alpha\beta + (\beta-\phi)\frac{\mu}{\gamma} + \mu > 0$$

$$m^* > \frac{\alpha\beta\left(1 - \frac{\mu}{\gamma}\right) - (\beta-\phi)\frac{\mu}{\gamma} - \mu}{2\alpha\beta}$$

$$m^* > \frac{1}{2}\left(1 - \frac{\mu}{\gamma} - \frac{\mu(\beta-\phi)}{\gamma\alpha\beta} - \frac{\mu}{\alpha\beta}\right) = -\frac{B}{2}$$

Therefore $C' > 0$ whenever $m^* > -\frac{B}{2}$.

We now try to tie together our findings to distinguish the two endemic equilibria. Having shown that $B' > 0$ and $C' > 0$ where $R_2 < 1$, we establish the necessary criteria for LAS for E_3 and E_4 where $m^* > -\frac{B}{2}$

Recalling that

$m^*_\pm =$

$$\frac{1}{2}\left[-\left[\left(\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1-\frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta}\right] \pm \sqrt{\left[\left(-\frac{\beta-\phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + \left(1-\frac{\mu}{\gamma}\right) - \frac{\mu}{\alpha\beta}\right]^2 + 4\left(\frac{\mu}{\gamma}\right)\left[\frac{\beta-\phi}{\alpha\beta}\left(1-\frac{\mu}{\gamma}\right) - \frac{\mu+\epsilon}{\alpha\beta}\right]}\right]$$

$$= \frac{1}{2}(-B \pm \sqrt{B^2 - 4C})$$

Since $m^*_+ > -\frac{B}{2}$, E_3 is LAS.
And since $m^*_- < -\frac{B}{2}$, then E_4 is unstable.

B Sensitivity Indices

Parameter	Parameter value	Sensitivity Index for R_1	Sensitivity Index for R_2
β	0.4	1.6	-9.6
ϕ	0.15	-0.6	3.6
μ	0.05	-.016	-0.16
ϵ	0.25	-0.83	-0.83
γ	0.20	0	1

Table 13: Sensitivity Indices for $R_2 < 0$.

Parameter	Parameter Value	Sensitivity Index for R_1	Sensitivity Index for R_2
β	0.2	4	-4.8
ϕ	0.15	-3	3.6
μ	0.05	-0.16	-0.16
ϵ	0.25	-0.83	-0.83
γ	0.20	0	1

Table 14: Sensitivity Indices for $R_2 < 0$.

Note above that we consider two β values to satisfy $R_2 < 0$. We do so in order to observe whether or not β remains the most sensitive parameter in the region $R_2 < 0$ which it does, as the relative indices suggest.

Parameter	Parameter Value	Sensitivity Index for R_1	Sensitivity Index for R_2
β	0.5	1.43	8.57
ϕ	0.15	-4.28	-2.57
μ	0.05	-0.16	-0.16
ϵ	0.25	-0.83	-0.83
γ	0.20	0	1

Table 15: Sensitivity Indices where $0 < R_2 < 1$.

Parameter	Parameter Value	Sensitivity Index for R_1	Sensitivity Index for R_2
β	0.6	1.3	2.6
ϕ	0.15	-0.3	-0.6
μ	0.05	-0.16	-0.16
ϵ	0.25	-0.83	-0.83
γ	0.20	0	1

Table 16: Sensitivity Indices for $R_2 > 1$.

C Ten Key Values of the Green Party[14]

1. **Grassroots Democracy**
   Every human being deserves a say in the decisions that affect their lives and not be subject to the will of another. Therefore, we will work to increase public participation at every level of government and to ensure that our public representatives are fully accountable to the people who elect them. We will also work to create new types of political organizations which expand the process of participatory democracy by directly including citizens in the decision-making process.

2. **Social Justice and Equal Opportunity**
   All persons should have the rights and opportunity to benefit equally from the resources afforded us by society and the environment. We must consciously confront in ourselves, our organizations, and society at large, barriers such as racism and class oppression, sexism and homophobia, ageism and disability, which act to deny fair treatment and equal justice under the law.

3. **Ecological Wisdom**
   Human societies must operate with the understanding that we are part of nature, not separate from nature. We must maintain an ecological balance and live within the ecological and resource limits of our communities and our planet. We support a sustainable society which utilizes resources in such a way that future generations will benefit and not suffer from the practices of our generation. To this end we must practice agriculture which replenishes the soil; move to an energy efficient economy; and live in ways that respect the integrity of natural systems.

4. **Non-Violence**
   It is essential that we develop effective alternatives to societys current patterns

of violence. We will work to demilitarize, and eliminate weapons of mass destruction, without being naive about the intentions of other governments. We recognize the need for self-defense and the defense of others who are in helpless situations. We promote non-violent methods to oppose practices and policies with which we disagree, and will guide our actions toward lasting personal, community and global peace.

5. **Decentralization**
   Centralization of wealth and power contributes to social and economic injustice, environmental destruction, and militarization. Therefore, we support a restructuring of social, political and economic institutions away from a system which is controlled by and mostly benefits the powerful few, to a democratic, less bureaucratic system. Decision-making should, as much as possible, remain at the individual and local level, while assuring that civil rights are protected for all citizens.

6. **Community Bases Economics and Economic Justice**
   We recognize it is essential to create a vibrant and sustainable economic system, one that can create jobs and provide a decent standard of living for all people while maintaining a healthy ecological balance. A successful economic system will offer meaningful work with dignity, while paying a living wage which reflects the real value of a persons work.

   Local communities must look to economic development that assures protection of the environment and workers rights; broad citizen participation in planning; and enhancement of our quality of life. We support independently owned and operated companies which are socially responsible, as well as co-operatives and public enterprises that distribute resources and control to more people through democratic participation.

7. **Feminism and Gender Equity**
   We have inherited a social system based on male domination of politics and economics. We call for the replacement of the cultural ethics of domination and control with more cooperative ways of interacting that respect differences of opinion and gender. Human values such as equity between the sexes, interpersonal responsibility, and honesty must be developed with moral conscience. We should remember that the process that determines our decisions and actions is just as important as achieving the outcome we want.

8. **Respect for Diversity**
   We believe it is important to value cultural, ethnic, racial, sexual, religious and

spiritual diversity, and to promote the development of respectful relationships across these lines.

We believe that the many diverse elements of society should be reflected in our organizations and decision-making bodies, and we support the leadership of people who have been traditionally closed out of leadership roles. We acknowledge and encourage respect for other life forms than our own and the preservation of biodiversity.

9. **Personal and Global Responsibility**
   We encourage individuals to act to improve their personal well-being and, at the same time, to enhance ecological balance and social harmony. We seek to join with people and organizations around the world to foster peace, economic justice, and the health of the planet.

10. **Future Focus and Sustainability**
    Our actions and policies should be motivated by long-term goals. We seek to protect valuable natural resources, safely disposing of or unmaking all waste we create, while developing a sustainable economics that does not depend on continual expansion for survival. We must counterbalance the drive for short-term profits by assuring that economic development, new technologies, and fiscal policies are responsible to future generations who will inherit the results of our actions.

D Solving for R_1' of the Two-Track Model

We calculate the reproductive number R_1' of the two track model using the next generation operator method where R_1' is analogous to R_1 of the one-track model. The party-free equilibrium is $(pN, (1-p)N, 0, 0, 0)$. After linearizing our system around the PFE and differentiating with respect to the system's variables V_H, V_L, and M, we formulate the following Jacobian matrix.

$$J = \begin{pmatrix} (\beta_1 - \phi_1\tau)\frac{H^*}{N} - (\mu + \epsilon_1 + \phi_1\frac{L^*}{N}) & \beta_1\frac{H^*}{N} & \alpha\beta_1\frac{H*}{N} - \gamma\frac{V_H^*}{N} \\ \beta_2\frac{L^*}{N} & (\beta_2 - \phi_2\tau)\frac{L^*}{N} - (\mu + \epsilon_2 + \phi_2\frac{H^*}{N} + \gamma\frac{M^*}{N} & \alpha\beta_2\frac{L^*}{N} - \gamma\frac{V_L^*}{N} \\ \gamma\frac{M^*}{N} & \gamma\frac{M^*}{N} & \gamma\frac{V_H^*+V_L^*}{N} - \mu \end{pmatrix}$$

Note that $J = M - D$, where the entries of M are nonnegative and D is a diagonal matrix. Separating J into M and D, we get the following matrices:

$$M = \begin{pmatrix} (\beta_1 - \phi_1\tau)\frac{H^*}{N} & \beta_1\frac{H^*}{N} & \alpha\beta_1\frac{H*}{N} \\ \beta_2\frac{L^*}{N} & (\beta_2 - \phi_2\tau)\frac{L^*}{N} + \gamma\frac{M^*}{N} & \alpha\beta_2\frac{L^*}{N} \\ \gamma\frac{M^*}{N} & \gamma\frac{M^*}{N} & \gamma\frac{V_H^*+V_L^*}{N} \end{pmatrix}$$

and

$$D = \begin{pmatrix} (\mu+\epsilon_1+\phi_1\frac{L^*}{N}) & 0 & \gamma\frac{V_H^*}{N} \\ 0 & (\mu+\epsilon_2+\phi_2\frac{H^*}{N}) & \gamma\frac{V_L^*}{N} \\ 0 & 0 & \mu \end{pmatrix}$$

We multiply MD^{-1} such that

$$MD^{-1} = \begin{pmatrix} \frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+\phi_1(1-p)} & \frac{(\beta_1 p)}{\mu+\epsilon_2+\phi_2 p} & \alpha\beta_1\frac{p}{\mu} \\ \frac{\beta_2(1-p)}{\mu+\epsilon_1+\phi_1(1-p)} & \frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+\phi_2 p} & \alpha\beta_2\frac{1-p}{\mu} \\ 0 & 0 & 0 \end{pmatrix}$$

The final step of the next generation method involves determining the maximum eigenvalue of the of the product MD^{-1} where the three eigenvalues are $0, \frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+\phi_1(1-p)}$, $\frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+\phi_2 p}$. We analytically solve for the characteristic polynomial and assign the maximum solution of the quadratic expression as follows:

$R_1' =$

$$\frac{1}{2}\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1} + \frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2} \pm \sqrt{\left[\frac{(\beta_1-\phi_1\tau)p}{\mu+\epsilon_1+(1-p)\phi_1} + \frac{(\beta_2-\phi_2\tau)(1-p)}{\mu+\epsilon_2+p\phi_2}\right]^2 + 4\frac{p\beta_1\beta_2(1-p)}{(\mu+\epsilon_1+(1-p)\phi_1)(\mu+\epsilon_2+p\phi_2)}}\right]$$

The expression above should, in theory, be the general version of the specific homogeneous one-track version. Therefore, we substitute in $p = 1$ and arrive at R_1:

For $p = 1$

$$R_1' = \frac{(\beta_1-\phi_1\tau)p}{\epsilon_1+\mu+\phi_1(1-p)}$$

where $\tau = 1$ since we are dealing with only one susceptible class; additionally, subscripts are unnecessary for the homogeneous case. After applying these simplifications R_1', as expected, reduces to $R_1 = \frac{\beta-\phi}{\epsilon+\mu}$.

E Explanation to graphic summary

We now go through the process of plotting figure 6. We first make the following substitutions:

$$y = \frac{\gamma}{\mu}$$

$$x = \frac{\alpha\beta}{\mu+\epsilon}$$

$$q = \frac{\beta - \phi}{\alpha\beta}$$

$$r = \frac{\mu}{\mu + \epsilon}$$

Note that $q < 1$ and $r < 1$. Making the appropriate substitutions, we derive an expression for the curve on which $R_1 = 1$:

$$R_1 = \frac{\beta - \phi}{\mu\epsilon} = \frac{\beta - \phi}{\alpha\beta}\frac{\alpha\beta}{\mu + \epsilon} = qx = 1$$

such that

$$x = \frac{1}{q}$$

Additionally, now we obtain an expression for the curve on which $R_2 = 1$:

$$R_2 = \frac{\gamma}{\mu}\left(1 - \frac{1}{\frac{\beta-\phi}{\mu+\epsilon}}\right) = 1$$

$$1 = y\left(1 - \frac{1}{qx}\right)$$

such that

$$y = \frac{qx}{qx - 1}$$

Note that this curve has a horizontal asymptote $y = 1$ and a vertical asymptote at $x = \frac{1}{q}$.

Now we find identify those curves for which $B = 0$ and $B^2 - 4C = 0$
Recall that

$$B = \left(\frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1 - \frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta},$$

$$C = -\frac{\mu}{\gamma}\left[\frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right) - \frac{\mu + \epsilon}{\alpha\beta}\right].$$

where, applying our substitutions $B = 0$ implies that $\frac{q}{y} - 1 + \frac{1}{y} + \frac{r}{x} = 0$
Therefore, we define the curve on which $B = 0$ as $y = \frac{x(q+1)}{x-r}$

Note that this curve has a horizontal asymptote at $y = q + 1$ and a vertical asymptote at $x = r$.

Additionally, $B^2 - 4C = 0$ implies that $\left(\frac{q}{y} - 1 + \frac{1}{y} + \frac{r}{x}\right)^2 + \frac{4}{y}\left(q\left(1 - \frac{1}{y}\right) - \frac{1}{x}\right) = 0$ such that the curve on which $B^2 - 4C = 0$ is redefined as

$$y = \frac{-x(qx + r - x - 2 + qr + 2\sqrt{(qr-1)(qx + r - 1 - x)})}{(r-x)^2}$$

Note that this curve has a vertical asymptote at $x = r$ and a horizontal asymptote at $y = 1 - q$.

Via the asymptotes of the redefined functions, we define regions for which $B < 0$, $B^2 - 4C > 0$, and $R_2 < 1$. We find that the region where $B^2 - 4C > 0$ is a subset of the region where $B < 0$. Therefore we only need to consider the region where $B^2 - 4C > 0$ in order to define the special region of backward bifurcation.
Applying the same method of using bounds to define regions of equilibria stability, we study the system's local dynamics.