

Two Mathematical Models for the Tympanic Membrane

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Abstract

In this project we present two mathematical models for the human tympanic membrane. The eardrum can be viewed as an example of the vibrating drum problem. In the first model, we treat the tympanic membrane as a rectangular region. In the second model the tympanic membrane is considered as a disk. Both models use a wave equations. We study the impact of changes in the membrane tension related to trauma or tumors which cause the frequencies of vibration to either increase or decrease.

1 Introduction

The human ear serves as a transducer, converting sound energy into mechanical energy and then mechanical energy into a nerve impulse which is transmitted to the brain. The ear's ability to do this allows us to perceive the pitch of sounds by detection of the wave's frequencies, the loudness of sound by detection of the wave's amplitude and the timbre of the sound by the detection of the various frequencies which make up a complex sound wave [10].

The ear consists of three basic parts - the outer ear, the middle ear, and the inner ear. Each part serves a specific purpose in detecting and interpreting sound. The outer ear collects and channels sound to the middle ear which transforms the energy of a sound wave into the internal vibrations of the bone structure of the middle ear and then transforms these vibrations into a compressional wave in the inner ear. The inner ear converts the energy of a compressional wave within the inner ear fluid into nerve impulses which are transmitted to the brain [10].

The outer ear consists of the ear flap and the ear canal. The outer ear channels sound waves to the tympanic membrane (eardrum) of the middle ear. The tympanic membrane converts the mechanical energy of the sound wave into vibrations which are conducted to the ossicles of the inner ear [10].

The middle ear is an air-filled cavity consisting of the tympanic membrane and the three ossicles which are called the malleus, incus and stapes. The tympanic membrane is a fibrous membrane which vibrates at the same frequency of the sound wave. The ossicles of the middle ear act as levers to amplify the vibrations of the sound wave. Due to a mechanical advantage, the displacements of the stapes are greater than that of the malleus. Furthermore, since the pressure wave striking the large area of the eardrum is concentrated into the smaller area of the stapes, the force of the vibrating stapes is nearly 15 times larger than that of the tympanic membrane [10].

The inner ear consists of the cochlea, the semicircular canals, and the auditory nerve. The cochlea and the semicircular canals are filled with endolymph. The endolymph and nerve cells serve as accelerometers for detecting accelerated movements and assisting in the task of maintaining balance. The cochlea is shaped like a helical spiral. Its inner surface is filled with fluid and lined with over 20 000 hairlike nerve cells which start moving if a compressional wave comes in from the interface between the malleus of the middle ear and the oval window of the inner ear through the cochlea.

Each cell has a given sensitivity to a particular frequency of vibration. When the frequency of the compressional wave matches the given frequency of the nerve cell, then the cell will resonate with a larger amplitude of vibration. This increased vibrational amplitude induces the cell to release an electrical impulse which is passed on by auditory nerve to the brain. In a process which is not clearly understood, the brain is capable of interpreting the qualities of the sound upon reception of these electric nerve impulses [10].

There are several causes of hearing loss, conductive hearing loss referring to the type of hearing loss caused by a mechanical problem in the outer or middle ear which might block the conduction of sound. Causes can include perforation of the tympanic membrane or tumors close to the tympanic membrane, the first of which may cause a reduction of the membrane's tension, the latter may cause an increase in tension [10].

There have been essentially two kinds of approaches in modelling the tympanic membrane, the first of which does not consider the ultrastructure of the membrane and a second one which accounts for the fibrous ultrastructure of the eardrum.

The first known model of the tympanic membrane was formulated by Helmholtz in 1873 [8]. He showed that sound waves cause the tympanic membrane to vibrate and that the ossicles conduct these vibrations to the inner ear. In 1941, Bekesy recorded the first measurements of the human eardrum and describes the tympanic membrane as a stiff plate [2]. In the 1979's, Tonndorf and Khanna used holographic experiments and a continuous model to study the restoring force within the tympanic membrane [14]. Laszlo and Funnell modeled the eardrum using a finite element method [6], ignoring the fibrous structure, but including several kinds of restoring forces. A continuous model which accounts for the fibrous ultrastructure of the tympanic membrane has been introduced by Rabbitt and Holmes in 1986 [12].

2 The General Model

We assume that the tympanic membrane is homogeneous and that one point on the membrane can move in only one direction.

The tympanic membrane has an approximately constant volume density ρ , somewhere between that of water ($1.0gcm^{-3}$) and that of undehydrated collagen ($1.2gcm^{-3}$) [7].

The tension T of the tympanic membrane has never been satisfactorily measured [6], therefore, we will restrict ourselves to a qualitative analysis regarding the tension parameter.

Let D be the region in the x, y -plane along whose boundaries the tympanic membrane is fastened, and let $u(x, y, t)$ denote the location of the point on the membrane with coordinates x, y at time t . Let ρ denote the constant density of the membrane [5].

Newton's second law describes the force F which is acting on a sufficiently small rectangular piece of D at (x, y) with sides dx, dy as

$$F = \rho \frac{\partial^2 u}{\partial t^2} (x, y) dx dy. \quad (1)$$

Let $u(x, y)$ be the initial position of the membrane. The force displaces it by $\delta(x, y)$ to

$$u(x, y) + \delta(x, y) \quad (2)$$

where $\delta(x, y)$ and its derivatives are assumed to be very small. Then the work performed on the region $dx dy$ by F is given by

$$F \delta(x, y) = \delta(x, y) \rho \frac{\partial^2 u}{\partial t^2} (x, y) dx dy \quad (3)$$

and the total work for the entire membrane is obtained by

$$\text{Work} = \int \int_D \delta(x, y) \rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy \quad (4)$$

The potential energy E_{pot} being stored in the membrane is directly proportional to the extent to which the membrane is stretched, where the stretching is approximated by

$$E_{pot} = \frac{T}{2} \int \int_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy \quad (5)$$

where T denotes the tension of the membrane which we assume to be constant. The new potential energy E_{pot}^{new} after displacement of the membrane by $\delta(x, y)$ can be determined as follows:

$$E_{pot}^{new} = \frac{T}{2} \int \int_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy + T \int \int_D \left[\frac{\partial u}{\partial x} \frac{\partial \delta}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \delta}{\partial y} \right] dx dy + \frac{T}{2} \int \int_D \left[\left(\frac{\partial \delta}{\partial x} \right)^2 + \left(\frac{\partial \delta}{\partial y} \right)^2 \right] dx dy \quad (6)$$

where the last term can be neglected if δ and its derivatives are sufficiently small.

Under this condition the difference in potential energy is

$$E_{pot}^{new} - E_{pot} = T \int \int_D \left[\frac{\partial u}{\partial x} \frac{\partial \delta}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \delta}{\partial y} \right] dx dy = \int \int_D T \nabla u \nabla \delta dx dy. \quad (7)$$

Now we apply Gauss' Theorem [5] in the plane to obtain

$$\int \int_D T \nabla u \nabla \delta dx dy + \int \int_D T \delta \nabla \nabla u dx dy = \int_{\partial D} T \delta \nabla u N ds. \quad (8)$$

However, δ vanishes on ∂D , and therefore, the change in potential is given by

$$- \int \int_D \delta(x, y) T (\nabla \nabla u)(x, y) dx dy \quad (9)$$

Since the work performed is the negative of the potential, we have

$$\int \int_D \delta(x, y) \rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy = \int \int_D \delta(x, y) T (\nabla \nabla u)(x, y) dx dy, \quad (10)$$

which holds for all δ . Therefore we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} = T (\nabla \nabla u) \quad (11)$$

which is equivalent to

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \nabla \nabla u \quad (12)$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (13)$$

which is the wave equation in two dimensions, where $c^2 := \frac{T}{\rho}$. Therefore the boundary value problem to be solved is:

Find $u(x, y, t)$ such that

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
2. $u(x, y, t) = 0$ for $(x, y) \in \partial D$ (Dirichlet boundary condition)
3. $u(x, y, 0) = g_1(x, y)$ and $\frac{\partial u}{\partial t}(x, y, 0) = g_2(x, y)$ (initial conditions)

where D is a bounded region in the (x, y) -plane, ∂D is piecewise smooth and $g_1, g_2 : D \rightarrow \mathbb{R}$ are continuous.

In the following two sections, the tympanic membrane is considered as a vibrating drum and this boundary value problem is solved for a rectangular and a circular region, thus allowing for two models of the tympanic membrane.

3 Solution Of The Rectangular Model

We now approximate the tympanic membrane by a rectangular region

$$D := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y \leq b\} \quad (14)$$

where the dimensions of the membrane are estimated by $a = b = 9mm$ [6]. Now the boundary condition becomes

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \quad (15)$$

The solution of the boundary value problem using a separation of variables ansatz is

$$u_{mn}(x, y, t) = \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right) [A \cos(\omega_{mn}t) + B \sin(\omega_{mn}t)], \quad m, n \in \mathbb{N}_0 \quad (16)$$

where

$$\omega_{mn} := c\sqrt{-\lambda_{mn}} \quad , \quad \lambda_{mn} := -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \quad (17)$$

The general solution is found as a superposition [5] of these solutions

$$u(x, y, t) = \sum_{m,n=1}^{\infty} \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right) [A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t)] \quad (18)$$

and A_{mn} and B_{mn} are determined from the initial conditions. Assuming that

$$u(x, y, 0) = u_0(x, y) \quad , \quad u_t(x, y, 0) = v_0(x, y) \quad (19)$$

we obtain

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b u_0(x, y) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right) \quad (20)$$

and

$$B_{mn} = \frac{4}{ab} \int_0^a \int_0^b v_0(x, y) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right) \quad (21)$$

The motion of the membrane is the superposition of infinitely many modes, where the mode $u_{mn}(x, y, t)$ oscillates with the frequency

$$s_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{1}{2\pi} c\sqrt{-\lambda_{mn}} = \frac{1}{2\pi} \sqrt{\frac{T}{\rho}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad (22)$$

We can see that if the tension T of the tympanic membrane increases, the frequency of vibration will increase. This suggests that in case of a perforation of the tympanic membrane which causes a decrease in tension, the frequencies of vibration decrease. As high frequency vibrations of the tympanic membrane are associated with high-frequency hearing, we would expect a high-frequency hearing loss in patients suffering from tympanic membrane perforations. Similarly, if a tumor is located in the middle ear that applies pressure to the ossicles, and therefore to the tympanic membrane, its tension increases, and we expect an increase in vibration frequencies, and thus a low-frequency hearing loss.

Remark: As we can see from the model, if the density ρ of the membrane increases, the frequency of vibration will decrease. However, the density of the human tympanic membrane appears to be rather constant and does not change with age or due to injuries [10].

4 Solution Of The Circular Model

After changing to polar coordinates by using the identities $x = r \cos \theta$ and $y = r \sin \theta$, where r is the radius and θ denotes the angle, we can restate the boundary value problem as:

For $0 < r \leq 1$, $\theta \in \mathbb{R}$, find $u(r, \theta)$ such that

1. $\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$
2. $u(r, \theta + 2\pi) = u(r, \theta)$ (periodicity)
3. u is "well-behaved" near $r = 0$
4. $u(1, \theta) = h(\theta)$, h "well-behaved" with $h(\theta + 2\pi) = h(\theta)$ (boundary condition)

Using separation of variables, the solution to the boundary value problem is

$$u_n(r, \theta) = a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta), \quad n \in \mathbb{N}_0. \quad (23)$$

Therefore, the general solution which satisfies the periodicity condition is

$$u(r, \theta) = \frac{a_0}{2} + \sum_1^{\infty} (a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)), \quad (24)$$

where $a_0, a_1, \dots, b_1, \dots$ are constants which can be determined by applying the boundary conditions:

$$h(\theta) = u(1, \theta) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad (25)$$

i.e. the constants $a_0, a_1, \dots, b_1, \dots$ are the Fourier coefficients of h .

We will see that in this model, determining the vibration frequencies and amplitudes reduces to solving the eigenvalue problem for the Laplace operator [5].

Let

$$D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}, \quad R \in \mathbb{R}_+ \quad (26)$$

denote the disk with radius R . Using separation of variables, $u(x, y, t) = f(x, y)g(t)$, leads to two eigenvalue problems

$$g''(t) = \lambda g(t) \quad (27)$$

and

$$\Delta f(x, y) = \lambda f(x, y) \quad (28)$$

where $\lambda \in \mathbb{R}$ is a constant and f must satisfy the boundary condition

$$f(x, y) = 0 \quad \text{for } (x, y) \in \partial D. \quad (29)$$

The expression in (28) is the eigenvalue problem for the Laplace operator. We shall see that its eigenvalues determine the frequencies of vibration of the tympanic membrane.

We use polar coordinates to compute the eigenvalues of the disk:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = \lambda f, \quad f(r, \theta) = 0 \quad \text{on } \partial D \quad (30)$$

Again, use separation of variables and let $f(r, \theta) = \varphi(r)\psi(\theta)$, $\varphi(R) = 0$, $\psi(\theta + 2\pi) = \psi(\theta)$, to obtain two ordinary differential equations

$$\frac{\partial^2 \psi}{\partial \theta^2} = \mu \psi, \quad \psi(\theta + 2\pi) = \psi(\theta) \quad (31)$$

and

$$r \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \lambda r^2 \varphi = -\mu \varphi, \quad \varphi(R) = 0 \quad (32)$$

where $\mu \in \mathbb{R}$ is constant. The only nontrivial solutions of (31) are

$$\mu = 0, \quad \psi = \frac{a_0}{2} \quad (33)$$

and

$$\mu = -n^2, \quad \psi = a_n \cos(n\theta) + b_n \sin(n\theta) \quad (34)$$

for $n \in \mathbb{N}$. Substitution into (32) and defining $x := \sqrt{-\lambda r}$ and $y := \varphi$ leads to

$$x \frac{\partial}{\partial x} \left(x \frac{\partial y}{\partial x} \right) + (x^2 - n^2)y = 0, \quad y(\sqrt{-\lambda}R) = 0 \quad (35)$$

which is Bessel's equation. For each $n \in \mathbb{N}$, (35) has a one-dimensional space of solutions which are constant multiples of the Bessel functions of the first kind J_n .

At this point, the eigenvalue problem of the Laplace operator amounts to the study of zeros of the Bessel functions [5]: the boundary condition $y(\sqrt{-\lambda}R) = 0$ requires $\sqrt{-\lambda}R$ to be a zero of J_n .

Let $x_{n,k}$ denote the k -th positive root of J_n . Then

$$\sqrt{-\lambda}R = x_{n,k} \quad (36)$$

implies

$$\lambda = -\frac{x_{n,k}^2}{R^2} \quad (37)$$

and therefore

$$\varphi(r) = J_n \left(\frac{x_{n,k}r}{R} \right) \quad (38)$$

is a solution of (35) which vanishes at $r = R$. For $n \in \mathbb{N}$,

$$\lambda_{n,k} = -\frac{x_{n,k}^2}{R^2} \quad (39)$$

is an eigenvalue and

$$f_{n,k} = J_n \left(\frac{x_{n,k}r}{R} \right) \cos(n\theta) \quad (40)$$

or

$$h_{n,k} = J_n \left(\frac{x_{n,k}r}{R} \right) \sin(n\theta) \quad (41)$$

are solutions to the eigenvalue problem. $f_{n,k}$ and $h_{n,k}$ are called eigenfunctions.

The general solution again will be a superposition of these modes of vibration.

To solve the initial value problem, we substitute the eigenvalues $\lambda_{n,k}$ into the equations for g and obtain

$$g(t) = K \sin(c\sqrt{-\lambda}t), \quad c = \sqrt{\frac{T}{\rho}}, \quad K \in \mathbb{R} \quad (42)$$

as solutions.

Each eigenfunction corresponds to one mode of oscillation of the tympanic membrane, i.e. the n,k -th mode vibrates with frequency

$$s_{n,k} = \frac{1}{R} \frac{c \sqrt{-\lambda_{n,k}}}{2\pi} = \frac{c}{R^2} \frac{x_{n,k}}{2\pi} = \frac{\sqrt{T}}{R^2} \frac{x_{n,k}}{2\pi}. \quad (43)$$

Thus we can see, as in the previous model, that the frequencies of vibration of the membrane is directly proportional to the tension of the membrane and indirectly proportional to its density. Therefore we obtain the same results as before regarding the impact of membrane perforation and tumors which are located close to the eardrum.

Here, the frequencies are also related to the radius R of the membrane, and the relation is quadratic. This implies that as the radius of the membrane decreases, the frequencies of vibration increase quadratically. This suggests that the modes of vibration of the tympanic membrane of a child oscillate at higher frequencies than the adult one: a child's tympanic membrane of radius R vibrates at modes of frequencies four times as great when compared to an adults tympanic membrane of radius $2R$. We can conclude that this may partly explain (apart from degeneration of the basilar membrane with age) why children have better high-frequency hearing than adults.

5 Conclusions

We have treated the tympanic membrane as a vibrating drum and used wave equations in both models. From the solutions to the rectangular model we can see that the tension of the membrane is directly proportional to the frequencies of vibration. Thus whenever there is an increase in tension, for example caused by a tumor which is located in the middle ear, the frequencies of vibration increase. Whereas, a reduced tension of the tympanic membrane, such as caused by a perforation, leads to a decrease in frequencies. When tumors of the middle ear and perforation of the

tympanic membrane are studied from a medical point of view, all of them mention hearing loss as a symptom, however, they do not specify a high or low-frequency hearing loss or the impact on the frequencies of vibration of the eardrum. It would be interesting to compare our results to clinical data from patients in order to see if this is in accordance with our findings.

From the solutions of the circular model we can see that the radius R of the membrane is inversely proportional to the frequency of its oscillation. Moreover, if the radius of the membrane is doubled, the frequencies of vibration decrease by a factor of 4. Therefore a child's tympanic eardrum of radius R vibrates at higher frequencies when compared to an adult's tympanic membrane of radius $2R$.

An idea for further work is to use classical plate theory and include elasticity theory into the model to account for the fibrous ultrastructure of the membrane. This will result in a fourth order partial differential equation model which will explain more aspects of the vibrating tympanic membrane, however, it will also be more complicated to analyze.

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