Mathematical Modeling of the Sex Worker Industry as a Supply and Demand System

Lily Davidoff[1], Karyn Sutton[2], Genevieve-Yvonne Toutain[2]
Fabio Sánchez[3], Christopher Kribs-Zaleta[4], Carlos Castillo-Chávez[2]
[1] Department of Mathematical Sciences,New Jersey Institute of Technology, Newark, NJ
[2] Department of Mathematics and Statistics, Arizona State University, Tempe, AZ
[3] Department of Biological Statistics and Computation Biology, Cornell University, Ithaca, NY
[4] Department of Mathematics, University of Texas at Arlington, Arlington, TX

Abstract

Prostitution is an occupation of global presence, often referred to as the worlds oldest, having existed for millennia. In the United States, the estimated annual prevalence of full-time sex workers is approximately 23 per every 100,000 individuals in the population [15]. We construct two mathematical models to explore the dynamics of the sex industry: one for the males who provide demand and another for the females who provide the supply. We perform qualitative analysis on these models separately, and explore the coupled system numerically. Through this analysis, we provide possible explanations as to why the current system of arrest and detainment does little to control the sex worker population. In addition, we show that if the efforts of legal enforcement focus on making male arrests, it is possible to significantly reduce the number of women in prostitution.

1 Introduction

Although prostitution is primarily illegal in the United States, with a few exceptions, escort services, strip bars, and other related localities, many of which front prostitution, are prevalent. In 1997, the Manhattan Yellow Pages contained 35 pages of escort services, and 52 pages in 1998. Such evidence suggests that prostitution is a growing occupation in cities in the United States such as Manhattan, New York [17].

The social infrastructure and the socioeconomic standing of a given location are highly related to the activity of its sex work industry [12]. Furthermore, females generally resort to prostitution out of economic need and sometimes out of familial obligation. Once females enter into prostitution, they often discover that it is difficult to leave the occupation. Frequently, the women discover that escape is not possible and resort to alcohol and drugs. Their addictions distract them emotionally from the reality of their situations [5]. Due to their previous occupation as sex workers, some women and girls who decide to leave the sex industry often find it difficult to find jobs elsewhere [4].

Looking at the sex industry as a market of supply and demand, where females are the primary supply and the males the main demand. Studies on pathways into prostitution observe that child abuse [14], sexual abuse [2], running away from home (homelessness) [19], [2], family breakdown, poverty, educational underachievement, unemployment and peer pressure are key risk factors that significantly increase a females odds of entering the sex industry [2]. In a particular study, while running away during childhood significantly increased an adolescents likelihood of entering into prostitution, childhood sexual abuse almost doubled that probability [19]. Previous studies have shown that male customers are of all nationalities and of all races. They come from a variety of socioeconomic backgrounds, and are generally between the ages of 15-90 years old. Only a small proportion of any given male population would be considered preliminary abstainers upon sexual maturity. Furthermore, women participating in a U.S. study observed that approximately 70-90% of the male customers they encountered were currently married [16]. Male customers generally ranged from working class men to professional men (lawyers, doctors etc.). A number of the women the U.S. study reported that police officers or undercover cops had asked for sex in exchange for dropping charges against them, and that some police are regular customers [16].

In the U.S. the average number of male customers women are expected to service per day is between 1 and 10; however, law enforcement estimates this average to be between 6 and 20 plus men in a day. When services such as stripping are included in this average, the average rises to 50 men per night. This increase in customers may be due to the fact that services such as stripping may allow for multiple simultaneous customers. In venues where costs are lower, women may be expected to have sex with 20 to 30 men in one day. Furthermore, at bachelor parties or conventions hired women might be expected to have sex with up to 20 men [16].

Few studies have examined the male demand as a driving force for the sexual exploitation of women and children in prostitution [16]. Swedish legislation recognizes that in the absence of male demand the number of female and girl commercial sex workers would decrease significantly. Without male customers the market of prostitution would cease to exist [3]. Furthermore, over the last ten years, men in countries within Asia and Africa have shown more interest in younger women and girls in prostitution. As a result of this demand, the average age at which girls enter the sex industry has decreased [19]. Moreover, in countries where prostitution has been legalized demand has continued to grow. In Victoria, Australia, with the legalization of prostitution, more brothels have emerged resulting in more men frequenting larger brothels. A study has shown that approximately 60,000 men spend around 7 million on prostitution weekly. In 1995, New South Wales legalized prostitution. By 1999 the current brothels had not only enlarged, but the quantity had tripled in size [16]. In the United States, where prostitution is primarily illegal, prostitution related arrests are predominantly female arrests, and sentences served are on average ineffectual. In Richmond Virginia, a police report states that of the 249 prostitution related arrests (male and female) made between July 1, 2001 and June 30, 2002, the median sentence assigned was three months; however, many of the sentences were suspended, resulting in a mean jail penalty of zero days [3].

In this paper we study the commercial sex industry in the United States as a market of supply and demand. Evidence supports our assumption that the male demand for sex as a commodity drives the sex industry; that regardless of the female sex worker supply, male demand is always satisfied. Furthermore, evidence suggests that legalization of prostitution is not improving the current situations in other countries. Hence, legal enforcement seems necessary in order to curb the demand for sex as a commodity. Moreover, the current mode of prostitution related enforcement in the United States has not proven to be successful at controlling the demand or the supply for prostitution. As a result, we would like to see if harsher punishments and other forms of intervention prove to be more effective. From such information one observes the importance of understanding the dynamics of the recruitment of women into prostitution, as well as the interplay between the male demand for sex as a commodity and the female supply of sex workers.

We construct a system of coupled mathematical models to describe the social dynamics of the sex industry as a market of supply and demand. In the male demand and female supply models, we treat recruitment into active customers, and into sex workers, as infectious diseases respectively. We analyze each system independently and couple them numerically. Overall, we will explore the effects of possible methods for reducing the female sex worker population by curbing male demand and by instituting potential rehabilitation programs.

2 Formulation of Prostitution Model

The interplay of commercial sex workers and their customers can be described as a supply and demand system. The male customers provide the demand for the sex worker industry, and the female commercial sex workers the supply. We consider a coupled model where males and females are represented as two distinct dynamical systems, coupled in the following way: the recruitment rate of commercial sex workers is driven by the demand of the male consumers, and similarly, the recruitment rate of customers is affected by the supply of commercial sex workers.

Equations (1) - (4) describe the dynamics of the women in the given population. The women who make up our system are those considered to be 'at risk' of becoming a commercial sex worker. Our model consists of at risk susceptible females, the S class, commercial sex workers, P, commercial sex workers detained in jail, J_f and those participating in rehabilitation programs, R. Assuming fluctuations in the population size are small compared to the total population, we take the female population to be constant. We assume that individuals born to females within the system at rate $\mu_f N$ are automatically 'at risk' for becoming commercial sex workers. Because we assume a constant population, the per capita exit rate is equal in magnitude to the per capita birth rate. Individuals exit the system naturally out of each class at rates $\mu_f S$, $\mu_f P$, $\mu_f J$, and $\mu_f R$. FUrthermore, we assume that females are involved in prostitution only between the age of 15-55 years old, thus we allow $\frac{1}{\mu_f} \approx 40$.

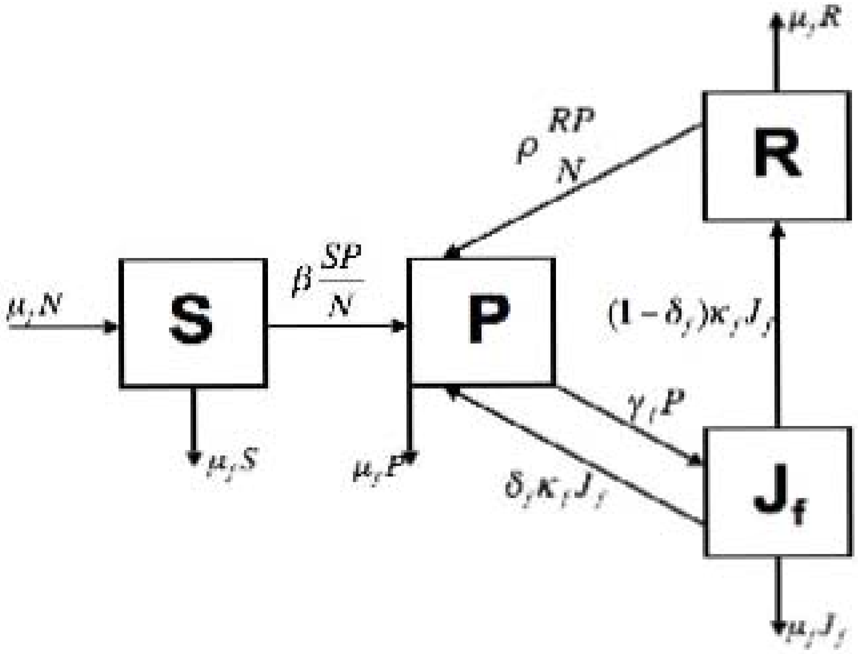

Figure 1: Caricature of the female model

$$\dot{S} = \mu_f N - \beta S \frac{P}{N} - \mu_f S \tag{1}$$

$$\dot{P} = \beta S \frac{P}{N} - (\gamma_f + \mu_f) P + \delta_f \kappa_f J_f + \rho R \frac{P}{N} \tag{2}$$

$$\dot{J}_f = \gamma_f P - (\kappa_f + \mu_f) J_f \tag{3}$$

$$\dot{R} = (1 - \delta_f) \kappa_f J_f - \rho R \frac{P}{N} + \mu_f R \tag{4}$$

In the female model S refers to the subpopulation of women who exhibit one or more of the following factors common among women who enter into prostitution: ran away from home during childhood [19], sexual or emotional abuse [1], poverty [12], and drug use [7]. These women may decide to enter into prostitution at a rate $\beta S \frac{P}{N}$, where β is a per capita recruitment rate and $\frac{P}{N}$ the chance of coming into contact with other commercial sex workers. Furthermore, P represents the population of commercial sex workers and N the total number of women considered in the system. Hence, the recruitment of women into prostitution is proportional to their contact with commercial sex workers. Studies show that many women working as sex workers had someone in their home or neighborhood

Parameter	Description of per capita rates	Value
β	recruitment rate of new commercial sex workers	1
γ_f	arrest rate of working commercial sex workers	1.37
ρ	relapse rate of rehabilitated commercial sex workers	3
κ_f	rate of leaving detention center	15
δ_f	proportion of detained commercial sex workers who immediately resume working	0.5

Table 1: Definition of Female Model Parameters

who was a sex worker as well [7], and that peer pressure plays a key role in the recruitment of females into prostitution [2].

Once working as commercial sex workers, the women are arrested at a rate $\gamma_f P$. Since greater police attention is paid to arresting sex workers when their prevalence increases [10], γ_f will have the form $\sigma_f \frac{P}{N}$. Because females are arrested primarily by the solicitation of undercover cops, and more rarely with an actual customer [8], we do not consider a direct relationship between the arrest rates of females and males. After serving their jail sentence, females may choose to go directly back to prostitution at a rate $\delta_f \kappa_f J_f$, or they may seek assistance and enter into rehabilitation at a rate $(1 - \delta_f)\kappa_f J_f$. The class characterized by rehabilitation of commercial sex workers, attempts to provide aid in establishing new lifestyles and career paths. While voluntary entry into rehabilitation programs is encouraged, it seems that most former sex workers in such programs enter as part of their sentence after being arrested [6], [11]. The rate of rehabilitation, therefore, could be viewed as a possible control parameter. Successful rehabilitation does not move individuals out of this class; however, the relapse rate $\rho\frac{RP}{N}$ moves individuals back to the commercial sex worker class.

In order to understand the females system we choose a set of realistic parameter values for the female model . We assume that each sex worker recruits about one new commercial sex workers a year, letting β be 1. Because each female sex worker gets arrested approximately 1.37 times per year, we let γ_f be 1.37. Furthermore we assume that rehabilitated individuals relapse back to prostitution at a rate $\rho = 3$, due to financial need and or peer influence. Arrested females spend on average anywhere between 0 days to 3 months in jail, thus, we let κ_f be 15 ($\frac{1}{15}$ of a year, approximately three weeks). Assuming that 50% of the females relapse directly back into prostitution while the other 50% attend rehabilitation programs we allow δ_f to be 0.5.

The male consumer process is represented by equations (5) - (8). In contrast to the subpopulation of women involved in prostitution, all sexually mature men between the ages of 15-55 are considered in this model. Males are assumed to leave the population as a result of maturing out of the sexually mature age group in which this type of activity is likely. Again, we consider fluctuations in population size to be small, and we assume a constant male population size, M. Within the model, men are characterized as potential commercial sex worker customers, C_p, those who would never patronize, or no longer will patronize sex workers G, active customers C_a, and those detained in jail, J_m. Again, men leave the system at rates $\mu_m C_p$, $\mu_m C_a$, $\mu_m G$, and $\mu_m J_M$. Because some proportion of the male population would never visit a sex worker, for fear of catching a disease, and or for fear of a partner finding out [13], men enter the model as either potential customers, at a rate $\Lambda\mu_m M$, or as permanent abstainers, at a rate $(1-\Lambda)\mu_m M$.

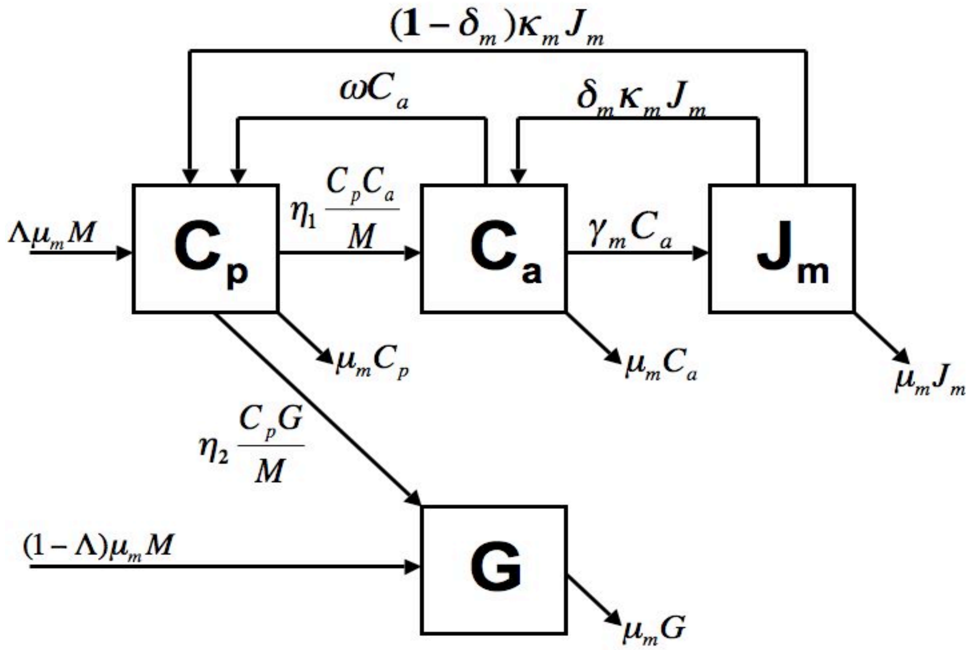

Figure 2: Caricature of the male model

$$\dot{C}_p = \Lambda\mu_m M - \eta_1 C_p \frac{C_a}{M} - \eta_2 C_p \frac{G}{M} + \omega_0 C_a + (1-\delta_m)\kappa_m J_m - \mu_m C_p \tag{5}$$

$$\dot{G} = (1-\Lambda)\mu_m M + \eta_2 C_p \frac{G}{M} - \mu_m G \tag{6}$$

$$\dot{C}_a = \eta_1 C_p \frac{C_a}{M} + \delta_m \kappa_m J_m - (\gamma_m + \omega_0 + \mu_m) C_a \tag{7}$$

$$\dot{J}_m = \gamma_m C_a - (\kappa_m + \mu_m) J_m \tag{8}$$

Visiting sex workers has historically been a social practice, especially in countries such as Thailand. It is not uncommon for men to take their sons to brothels, and business men are encouraged to hold meetings in bars and cafes where naked women dance. Furthermore, on occasion sex workers might be expected to have sex with up to 20 men if hired for bachelor parties or conventions [16]. Thus, potential customers are recruited to active customers through social interactions with existing customers at a rate $\eta_1 C_p \frac{C_a}{M}$. On the

Parameter	Description of per capita rates	Value
Λ	proportion of the male population that mature to be potential customers	0.9
η_1	recruitment rate of potential to active customers	30
η_2	recruitment rate of potential customers to abstainers	0.25
γ_m	arrest rate of active customers	0.25
ω	rate of ceasing activity due to knowledge of arrests	0.5
κ_m	rate of leaving jail/detention center	15
δ_m	proportion of detained customers that immediately resume activity	0.7

Table 2: Definition of Male Model Parameters

other hand, potential customers may decide to never visit commercial sex workers abstain as a result of social interaction with others who find it unacceptable, at a rate $\eta_2 C_p \frac{G}{M}$. Active customers are arrested at a rate $\gamma_m C_a$, and typically by female officers posing as sex workers [8]. Furthermore, legal enforcement is escalated in response to increasing activity of customers [10], therefore we take γ_m to be $\gamma_m = \frac{\sigma_m C_a}{M}$. Also, active customers may temporarily be dissuaded from visiting commercial sex workers because of hearing about increased legal enforcement, at some rate ωC_a. If this occurs primarily through word of mouth ω will take the form $\omega_0 \frac{J_m}{C_a+J_m+1}$. However, if knowledge of increased legal enforcement of commercial sex worker customers spreads via the mass media, ω will be simply proportional to the number of those in jail, $\omega_0 J_m$. Upon release from jail, men may either temporarily stop using commercial sex workers, at some rate $(1-\delta_m)\kappa_m J_m$, or they may return immediately to being active commercial sex worker customers at a rate $\delta_m \kappa_m J_m$. We assume that customers only temporarily cease actively using commercial sex workers, and then become susceptible to recruitment either into the C_a or G class. Similar to a recovering alcoholic, or someone who has quit smoking, the success of their rehabilitation is assumed to depend on their social interactions.

We choose a set of reasonable parameter values for which R_m is greater than one . We assume that the majority of the male population is susceptible to becoming active customers, thus, we let Λ be 0.9. We assume that each active customer influences approximately 30 potential customers to become active customers per year, letting $\eta_1 = 30$, and we assume that each abstainer influences less than 0.25 individuals to join the G class per year. We let γ_m be 0.25 since each active customer gets arrested approximately 0.25 times a year, and we let ω be 0.5, because we assume that less than 1 individuals will stop patronizing females. The average amount of time that jailed individuals serve is somewhere between 0 days to 3 months, hence, we let κ_m be 15 ($\frac{1}{15}$ of a year, approximately three weeks). We choose δ_m to be 0.7, assuming that a larger proportion of males released from jail will immediately relapse back into the active customer class rather than changing their lifestyles temporarily.

The dynamics of the male system impacts the female recruitment rates into prostitution, β and ρ, by providing the demand for these services. Without demand for sex workers, the activity in the industry would cease to exist [16]. High demand alone, however, is not enough to ensure high recruitment - if there are a relatively high number of commercial sex workers working, the profits earned by each is low compared to if there are relatively few commercial sex workers working. Therefore, the demand per commercial sex worker, i.e., expected profits should impact the recruitment of new commercial sex workers from the S and R classes. We assume the forms $\beta = \beta_0 \frac{k_m C_a}{P}$ and $\rho = \rho_0 \frac{k_m C_a}{P}$ for these rates. Assuming that supply exceeds demand [3], the sexual activity of the female workers are determined by the activity desired by the males. Then for k_m, the maximum desired sexual activity of a typical customer, the sexual activity of each female is $\frac{k_m C_a}{P}$. In correspondence with current literature, $\rho_0 > \beta_0$, indicating that relapse of previous commercial sex workers is more common than the recruitment of new ones, [11]. One reason for this seems to be that the current rehabilitation methods are ineffective at best [4], and usually result in women returning to their previous occupations as sex workers. Sex workers frequently develop drug and/or alcohol dependencies as a result of their environments [5], which noticeably affect their chances for successful rehabilitation as well.

Although the number of sex workers in any given area is significantly smaller than the number of male customers,based on current reports of $20-30$ customers per day, each sex worker can service around $10,000$ customers per year. Presumably some portion of those are repeat customers, so $10,000$ is likely an overestimate. This ratio implies that small decreases in the sex worker pool would go relatively unnoticed by their customers, so we assume the form $\eta_1 = \frac{\eta_m P}{P+\epsilon N}$ where $\epsilon << 1$.

3 Qualitative Analysis of the Male Model

For the qualitative analysis of the male model we consider all model parameters to be constant for tractability. Since population size is assumed to be constant, we can take now our state variables as c_p, the proportion of potential customers in a male population, g the proportion of abstainers in a population, c_a the proportion of active customers, and j_m the proportion of detained individuals in a population. The reformulated model is shown below in equations (9) - (12). In the next section we explore the existence of and where possible, local stability of the equilibria of the male model. We discuss conditions under which a population without sex worker customers is stable, and a situation where a population of sex worker customers can be established.

$$\dot{c}_p = \Lambda\mu_m - \eta_1 c_p c_a - \eta_2 c_p g + \omega c_a + (1-\delta_m)\kappa_m j_m - \mu_m c_p \quad (9)$$

$$\dot{g} = (1-\Lambda)\mu_m + \eta_2 c_p g - \mu_m g \quad (10)$$

$$\dot{c}_a = \eta_1 c_p c_a + \delta_m \kappa_m j_m - (\gamma_m + \omega + \mu_m) c_a \quad (11)$$

$$\dot{j}_m = \gamma_m c_a - (\kappa_m + \mu_m) j_m \quad (12)$$

$\beta(\mathbf{C_a}, \mathbf{P})$	$\beta = \beta_0 \mathbf{k_m} \frac{\mathbf{C_a}}{\mathbf{P}}$
β_0	scaling factor $\frac{[\text{Prostitutes}]}{[\text{transaction}]}$
k_m	Maximum sexual activity desired by average male customer $\frac{[\text{transactions}]}{[\text{time}]}$
$\rho(\mathbf{C_a}, \mathbf{P})$	$\rho = \rho_0 \mathbf{k_m} \frac{\mathbf{C_a}}{\mathbf{P}}$
ρ_0	scaling factor $\frac{[\text{Prostitutes}]}{[\text{transaction}]}$
k_m	Maximum sexual activity desired by average male customer $\frac{[\text{transactions}]}{[\text{time}]}$
$\eta_1(\mathbf{P})$	$\eta_1 = \frac{\eta_\mathbf{m} \mathbf{P}}{\mathbf{P} + \epsilon \mathbf{N}}$
η_m	Maximum recruitment rate
ϵ	scaling factor for rate of saturation of supply
$\gamma_\mathbf{m}(\mathbf{C_a})$	$\gamma_\mathbf{m} = \frac{\sigma_\mathbf{m} \mathbf{C_a}}{\mathbf{M}}$
σ_m	scalar effect of prevalence of active customers on arrest rate
$\gamma_\mathbf{f}(\mathbf{P})$	$\gamma_\mathbf{f} = \frac{\sigma_\mathbf{f} \mathbf{P}}{\mathbf{N}}$
σ_f	scalar effect of prostitute prevalence on arrest rate
$\omega(\mathbf{J_m}, \mathbf{C_a}, \mathbf{C_p})$	$\omega = \frac{\omega_0 \mathbf{J_m}}{\mathbf{C_a} + \mathbf{C_p} + \mathbf{J_m}}$
ω_0	scaling factor for effect of detained customers on the rate of relapse back to C_p

Table 3: Parameters as Functions of State Variables

3.1 The Active Customer Free Equilibrium ($\mathbf{C_a}$FE) and $\mathcal{R}_0 = \mathcal{R}_m$ For the Male System

We begin by looking for an equilibrium without any active customers, hereafter referred to as the C_aFE and denoted by $C_a\text{FE} = (c_p^*, g^*, c_a^*, j_m)$.

$$C_aFE : (c_p^*, g^*, c_a^*, j_m^*) = (1 - g^*, g^*, 0, 0),$$

where $g^* = \frac{1}{2}\left(1 - \frac{\mu}{\eta_2} + \sqrt{\left(1 - \frac{\mu}{\eta_2}\right)^2 + 4\frac{\mu}{\eta_2}(1 - \Lambda)}\right)$.

Throughout the rest of the paper, we denote $\frac{C_p^*}{M} = c^*$. The lack of a subscript for c^* should not cause any confusion considering $C_a^* = 0$.

At this equilibrium, the entire male population resides in the c_p and g classes, representing a society devoid of sex worker patrons. While the C_aFE represents the ideal situation, this is not presently the case in most major U.S. cities. Therefore, it is useful to look for conditions under which this equilibrium is unstable, and a population of active customers may be established. We proceed then by looking for threshold quantities and their implications for the stability of the C_aFE.

In compartmental epidemiological models there usually exists one or more threshold quantities which can be used to characterize the stability and existence of equilibria. One threshold quantity, the basic reproductive number $\mathcal{R}_0$, typically represents the average number of secondary infections produced by a typical infective in a population of susceptible individuals. In the context of our model this translates to the average number of potential customers recruited to active by a typical active customer, in a population of mostly potential customers and abstainers. Typically, if $\mathcal{R}_0 > 1$ the C_aFE is unstable and a state where sex worker patronage is prevalent ($C_a > 0$) exists. However, if $\mathcal{R}_0 < 1$ the C_aFE is stable, and there is no equilibrium with $C_a > 0$. We can compute this reproductive number which we will call $\mathcal{R}_m$ for this model using the next generation operator method presented in [9].

We let $\mathbf{X} = (C_p, G)^T$, $\mathbf{Y} = j_m$, and $\mathbf{Z} = c_a$, since we are considering c_a to be the only infective class, with j_m a latent class. Then $\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, $\dot{\mathbf{Y}} = g(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, and $\dot{\mathbf{Z}} = h(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$. $g(\mathbf{X}^*, \mathbf{Y}, \mathbf{Z}) = 0$ implicitly determines $\mathbf{Y} = \tilde{g}(\mathbf{X}^*, \mathbf{Z})$ which for our system is $j_m = \frac{\gamma_m}{\mu_m+\kappa_m} c_a$.

$$\mathbf{A} = \mathbf{D_Z} h(\mathbf{X}^*, \tilde{g}(\mathbf{X}^*, \mathbf{0}), \mathbf{0}) = \eta_1 c^* + \delta_m \gamma_m \frac{\kappa_m}{\mu_m + \kappa_m} - (\omega + \gamma_m + \mu_m)$$

$\mathbf{A}$ can be written as $\mathbf{M}$ and $\mathbf{D}$, where $\mathbf{M} = \eta_1 c^* + \delta_m \gamma_m \frac{\kappa_m}{\mu_m+\kappa_m}$ and $\mathbf{D} = \omega + \gamma_m + \mu_m$, and $\mathcal{R}_m = \mathbf{M}\mathbf{D}^{-1}$.

$$\mathcal{R}_m = \frac{\eta_1 c^*}{\omega + \gamma_m + \mu_m} + \delta_m \frac{\gamma_m}{\omega + \gamma_m + \mu_m} \frac{\kappa_m}{\mu_m + \kappa_m}$$

Since c^* represents the proportion of the population that can be recruited to active customers, $\eta_1 c^*$ is the average number of potential customers recruited per active customer per unit time. $\frac{1}{\omega+\gamma_m+\mu_m}$ is the average length of time a customer is active. In addition, the fraction $\frac{\gamma_m}{\omega+\gamma_m+\mu_m}$ represents the fraction of individuals who get arrested before leaving the system, $\frac{\kappa_m}{\kappa_m+\mu_m}$ represents the fraction of arrested individuals who leave jail before leaving the system, and δ_m are the proportion of these two groups together that return to visiting prostitutes after being arrested. So $\mathcal{R}_m$ is the average number of potential customers recruited per active customer during the time that they are active and the fraction of the customers who resume visiting prostitutes after being arrested. The stability of the C_aFE is guaranteed by the results presented in [9], and also explicitly discussed in the appendix.

3.2 The $\mathbf{C_a}$ prevalent equilibrium, $\mathbf{C_a}$PE

Next we look for nontrivial equilibria of the system, that is, we look for an equilibrium with $\bar{c}_p > 0$, $\bar{g} > 0$, $\bar{c}_a > 0$, and $\bar{j}_m > 0$. We call the active customer prevalent equilibrium and denote it by $c_a\text{PE} = (\bar{c}_p, \bar{g}, \bar{c}_a, \bar{j}_m)$.

$$\bar{c}_p = K$$
$$\bar{g} = \frac{(1-\Lambda)}{1-\frac{\eta_2}{\mu_m}K}$$
$$\bar{c}_a = \frac{\Lambda\mu_m - \eta_2 K\left(\frac{(1-\Lambda)}{1-\frac{\eta_2}{\mu_m}K} - \frac{\mu_m}{\eta_2}\right)}{\left(1-\frac{\kappa_m}{\mu_m+\kappa_m}\right)\gamma_m + \mu_m}$$
$$\bar{j}_m = \frac{\gamma_m}{\kappa_m+\mu_m}\bar{c}_a$$

where $K = \frac{(\omega+\gamma_m+\mu_m)}{\eta_1} - \frac{\delta_m\gamma_m}{\eta_1}\frac{\kappa_m}{\kappa_m+\mu_m}$.

To investigate when the nontrivial equilibrium exists, we plot the steady state c_a as a function of $\mathcal{R}_m$ in figure 3, and we see that it is positive when $\mathcal{R}_m > 1$. For these parameters then we see that a population of active customers can become established (the C_aFE loses stability and the C_aFE is positive) when $\mathcal{R}_m > 1$.

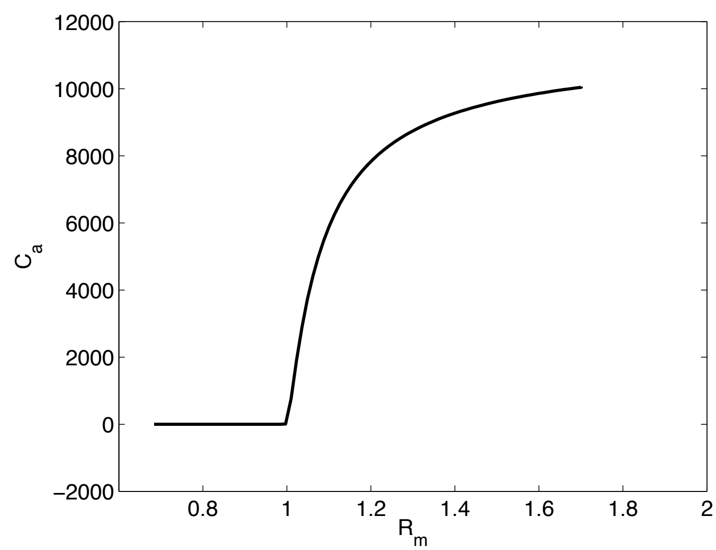

Figure 3: Existence of C_aPE

4 Qualitative Analysis of the Female Model

4.1 Prostitution Free Equilibrium (PFE) and $\mathcal{R}_0 = \mathcal{R}_f$ of the Female System

We begin the qualitative analysis of the female model with all constant parameters (β, ρ, γ_f constant). We first look for an equilibrium for the female model without prostitution, or with $P = 0$. We call this our Prostitution Free Equilibrium (PFE) and denote the PFE throughout by (S^*, P^*, J_f^*, R^*).

$$(S^*, P^*, J_f^*, R^*) = (N, 0, 0, 0).$$

As with the male model, we find the reproductive number for this system in an effort to gain insight into the dynamics of the system with the introduction of a sex worker in a mostly at risk population. Again, we use the next generation operator method as with the reproductive number for the males. We let $\mathbf{X} = (S, R)^T$, $\mathbf{Y} = J_f$, $\mathbf{Z} = P$. Now $\tilde{g}(\mathbf{X}^*, \mathbf{Z}) = J_f = \frac{\gamma_f}{\mu_f + \kappa_f} P$.

$$\mathbf{A} = \mathbf{D_z} h(\mathbf{X}^*, \tilde{g}(\mathbf{X}^*, \mathbf{0}), \mathbf{0}) = \beta - \gamma_f + \delta_f \gamma_f \frac{\kappa_f}{\mu_f + \kappa_f} - \mu_f$$

Similar to the male model,

$$\mathcal{R}_f = \mathbf{M}\mathbf{D}^{-1} = \frac{\beta}{\mu + \gamma_f} + \delta_f \frac{\gamma_f}{\mu_f + \gamma_f} \frac{\kappa_f}{\mu_f + \kappa_f}.$$

For this model, at the trivial equilibrium all females could possibly be recruited to the sex working class. So β is the average number of women in the 'at risk' population recruited to the population of women directly involved in prostitution by a typical sex worker in a year. The denominator in the first term, $\mu_f + \gamma_f$ is the reciprocal of the average time spent in the P class, i.e. the average time actively working in the sex industry. Additional recruits are from the return of arrested sex workers J_f to the active workers class, represented by the term $\delta_f \frac{\gamma_f}{\mu_f + \gamma_f} \frac{\kappa_f}{\mu_f + \kappa_f}$.

4.2 Prostitution Prevalent Equilibria, PPE

When solving for nontrivial equilibria of the female system, $(\bar{s}, \bar{p}, \bar{j}_f, \bar{r})$, we obtain expressions for all variables in terms of p. A quadratic polynomial in p results, suggesting the possibility of two positive equilibria. Furthermore, the structure of this model is similar to others that have exhibited backward bifurcation at $\mathcal{R}_0 = 1$ [18]. We find necessary conditions for a backward bifurcation to occur from the coefficients of the quadratic equation and using this as a starting point, we exhibit this behavior numerically. The dependence of the other state variables at the prostitution prevalent equilibria are shown below.

$$\bar{s} = \frac{\mu_f}{\beta\bar{p} + \mu_f}$$
$$\bar{j}_f = \frac{\gamma_f}{\kappa_f + \mu_f}\bar{p}$$
$$\bar{r} = \frac{(1-\delta_f)\kappa_f \frac{\gamma_f}{\kappa_f+\mu_f}\bar{p}}{\rho\bar{p} + \mu_f}$$

$\bar{p}_\pm$ are the positive solutions to the quadratic equation $aP^2 + bP + c = 0$, where

$$a = -\frac{\beta\rho}{\mu_f + \gamma_f}\left(1 + \frac{\gamma_f}{\kappa_f + \mu_f}\right)$$
$$b = \rho\left(\frac{\beta}{\gamma_f + \mu_f} + \frac{\gamma_f}{\gamma_f + \mu_f}\frac{\kappa_f}{\kappa_f + \mu_f} - 1\right) + \beta\left(\delta_f \frac{\gamma_f}{\gamma_f + \mu_f}\frac{\kappa_f}{\kappa_f + \mu_f} - 1\right)$$
$$c = \mu_f(\mathcal{R}_f - 1)$$

If $\mathcal{R}_f < 1$, $c < 0$. Thus, we have an upper bound for β: $\mathcal{R}_f < 1 \iff \beta < \mu_f + (1-\delta_f)\gamma_f \frac{\mu_f}{\mu_f+\kappa_f}$. Then a and c have the same sign if $\mathcal{R}_f < 1$ and $b > 0$ is also a necessary requirement for there to be two positive PPEs. The condition $b > 0$ can be written as

$$\beta > \mu_f + \gamma_f \frac{\mu_f}{\mu_f + \kappa_f} \tag{13}$$

and

$$\rho > (\mu_f + \gamma_f)\frac{\left(1 - \delta_f \frac{\gamma_f}{\mu_f+\gamma_f}\frac{\kappa_f}{\mu_f+\kappa_f}\right)}{\left(1 - \frac{\mu_f + \gamma_f \frac{\mu_f + (1-\delta_f)\kappa_f}{\mu_f+\kappa_f}}{\rho}\right)} \tag{14}$$

where (13) not only provides a lower estimate for β, but also ensures the positivity of the right hand side of (14). Numerically, we calculate a range of β that satisfies these requirements based on the standard parameter set. We note that the condition on ρ is automatically met. We check the sign of the discriminant $b^2 - 4ac$ and plot the solutions with positive discriminant as a function of $\mathcal{R}_f$. We see that there are two positive equilibria for $\mathcal{R}_f$ greater than ≈ 0.5.

If solutions to the system begin between the two positive equilibria $P_\pm$, they approach the greater of the two, P_+. An investigation of the dependence of solutions on initial conditions are shown in Figure **??**. Thus, there are many solutions to this system with $\mathcal{R}_f < 1$ that approach the Prostitution Prevalent Equilibrium, p_+ 5

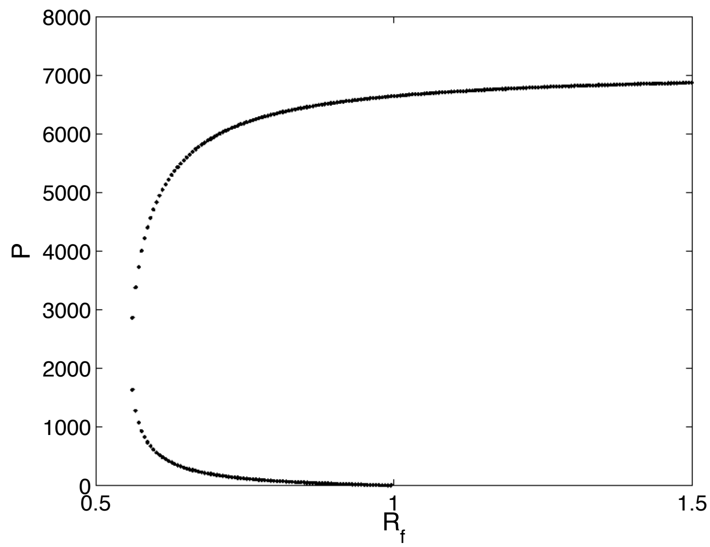

Figure 4: Nontrivial equilibria PPE for $\mathcal{R}_f < 1$

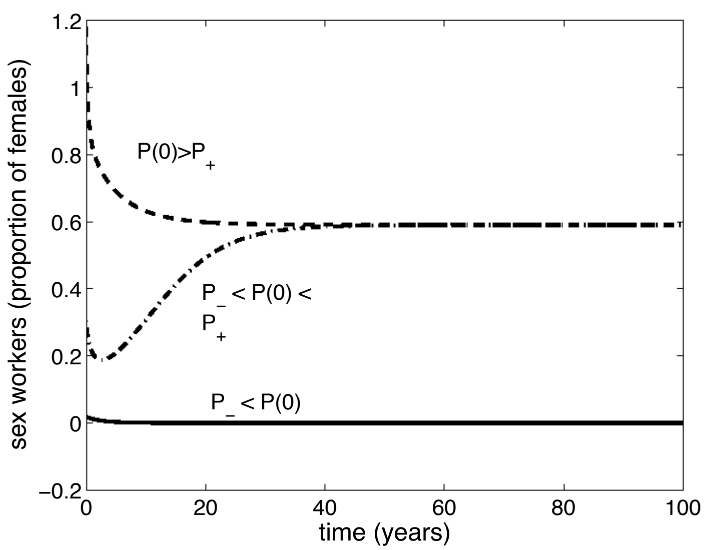

Figure 5: Investigation of the dependence of solutions on initial conditions in the region of bistability where $\mathcal{R}_f < 1$

Thus, in contrast to our findings in the male model, $\mathcal{R}_m < 1$ does not imply stability of the trivial equilibrium. However, we see that there is a threshold, $\mathcal{R}_c$ such that when

$\mathcal{R}_f = \mathcal{R}_c$ the two roots coincide, and thus the parameter regime with $\mathcal{R}_f < \mathcal{R}_c$ has the PFE as the only stable equilibrium. However, in practice it is not realistic to try to reduce $\mathcal{R}_f$ to below $\mathcal{R}_c$ as such a state is usually unobtainable. In the case of the sex worker industry, it is likely that $\mathcal{R}_f$ may not be reduced very much by additional efforts since efforts are currently focusing on this task. The implications of this result for possible control of the sex worker industry warrant further discussion. If it were possible to eliminate the sex worker population simply by decreasing this threshold below one, it would be effective to focus enforcement efforts primarily toward females. However, it is not likely that in a realistic situtation this will be possible, even with unlimited resources. Indeed, legal enforcement focuses on female arrests, and prostitution is established and prevalent in many U.S. cities. data Furthermore, it is possible to establish a population of sex workers in a population without prostitution and with seemingly good control measures, if it is introduced at a level above $\bar{p}_-$. This suggests that if we seek to disrupt this industry we must focus our attention elsewhere, thus motivating our inclusion of the male population in this study. Throughout the rest of the paper we explore the ways in which the model parameters affect the dynamics of the males and females separately and together.

5 Sensitivity Analysis of Reproductive Numbers

We conduct sensitivity analysis to observe how R_m and R_f change in relation to changes in each of the parameters in their respective models. We determine the normalized forward sensitivity indices, $S_p := \frac{\partial R_m}{\partial p}\frac{p}{R_m}$ and $S_p := \frac{\partial R_f}{\partial p}\frac{p}{R_f}]$, for p, parameter.

5.1 Sensitivity of R_m to model parameters

We find the sensitivity indices of the parameters in the male model to be

$S_{R_m}(\eta_1) = \frac{\eta_1}{R_m}\frac{c^*}{\omega+\gamma_m+\mu_m}$,
$S_{R_m}(\omega) = -\frac{\omega}{\omega+\gamma_m+\mu_m}$,
$S_{R_m}(\gamma_m) = \frac{\gamma_m}{\omega+\gamma_m+\mu_m}(\frac{\delta_m}{R_m}\frac{\kappa_m}{\kappa_m+\mu_m} - 1)$,
$S_{R_m}(\kappa_m) = \frac{\delta_m\gamma_m\kappa_m\mu_m}{(\kappa_m+\mu_m)^2}(\frac{1}{\eta_1 c^*+\delta_m\gamma_m\frac{\kappa_m}{\kappa_m+\mu_m}})$,
$S_{R_m}(\delta_m) = \delta_m\gamma_m\kappa_m(\frac{1}{\eta_1 c^*+\delta_m\gamma_m\frac{\kappa_m}{\kappa_m+\mu_m}})$, and
$S_{R_m}(\mu_m) = -\frac{1}{(\omega+\gamma_m+\mu_m)^2}(\eta_1 c^* + \frac{\delta_m\gamma_m\kappa_m}{\mu_m+\kappa_m}) + \frac{1}{\omega+\gamma_m+\mu_m}(\eta_1\frac{\partial c^*}{\partial\mu_m} - \frac{\delta_m\gamma_m\kappa_m}{(\mu_m+\kappa_m)^2})$,
such that
$\frac{\partial c^*}{\partial\mu_m} = \frac{1}{2\eta_2} - \frac{1}{4}\sqrt{(1-\frac{\mu_m}{\eta_2})^2 + 4\frac{\mu_m}{\eta_2}(1-\Lambda)}(2(1-\frac{\mu_m}{\eta_2})(-\frac{1}{\eta_2}) + 4\frac{1}{\eta_2}(1-\Lambda))$.

Using the standard parameter set, we compute these values to be

.

We observe a large positive index for δ_m, however, or the entire parameter space of δ_m, R_m is always greater than one. Thus we conclude that R_m is most sensitive to changes in η_1 and ω. While we can't directly change omega, perhaps we can change the rate at which individuals are arrested and drive R_m to be less than one. Thus in Figure ..., we

$S_{\eta_1}(R_m)$	.7938
$S_\omega(R_m)$	-0.6557
$S_{\gamma_m}(R_m)$	-0.1217
$S_{\kappa_m}(R_m)$	$1.7166e-4$
$S_{\delta_m}(R_m)$	3.00950

Table 4: Male Sensitivity

$S_\beta(R_f)$	4.2988
$S_{\gamma_f}(R_f)$	$-.0576$
$S_{\mu_f}(R_f)$	$-.07439$
$S_{\delta_f}(R_f)$	0.5917
$S_{\kappa_f}(R_f)$	0.0102

Table 5: Female Sensitivity

graph R_m as a function of ω and γ_f. We find that for small changes in the dissuasion rate (ω) when combined with a responsive arrest rate we can obtain $R_m < 1$.

5.2 Sensitivity of R_f to model parameters

We find the sensitivity indices of the parameters in the female model to be $S_{R_f}(\beta) = \frac{1}{(\mu_f+\gamma_f)^2}(1+\delta_f\frac{\gamma_f}{\beta}\frac{\kappa_f}{\mu_f+\gamma_f})$,

$S_{R_f}(R\gamma_f) = \frac{\gamma_f}{\mu_f+\gamma_f}(\frac{1}{R_f}\frac{\delta_f\kappa_f}{\kappa_f+\mu}-1)$,

$S_{R_f}(\mu_f) = -\frac{1}{\mu_f+\gamma_f}(1+\frac{\delta_f\gamma_f}{R_f}\frac{\kappa_f}{(\kappa_f+\mu_f)^2})$,

$S_{R_f}(\delta_f) = \frac{\delta_f}{R_f}\frac{\gamma_f}{\mu_f+\gamma_f}\frac{\kappa_f}{\kappa_f+\mu_f}$,

$S_{R_f}(\kappa_f) = \frac{\kappa_f}{R_f}\frac{\delta_f\gamma_f}{\mu_f+\gamma_f}\frac{\mu_f}{\mu_f+\kappa_f}$.

Using the standard parameter set for the female model
we obtain .

For comparable changes in parameter values, the relative changes in R_f are exhibited in the table above. Larger values for S_p result in greater changes to R_f for the same change in parameter value. This means that with small changes in $/beta$, the resulting change in R_f will be eight times greater than the resulting change in R_f due to a small change in $delta_f$. Thus for the female model, β is most sensitive.

6 Numerical Solutions of the Coupled Model

To explore the effects of the male and female systems on each other we proceed numerically. In Figure 6, we look at the effects of the threshold values of each model on the coupled

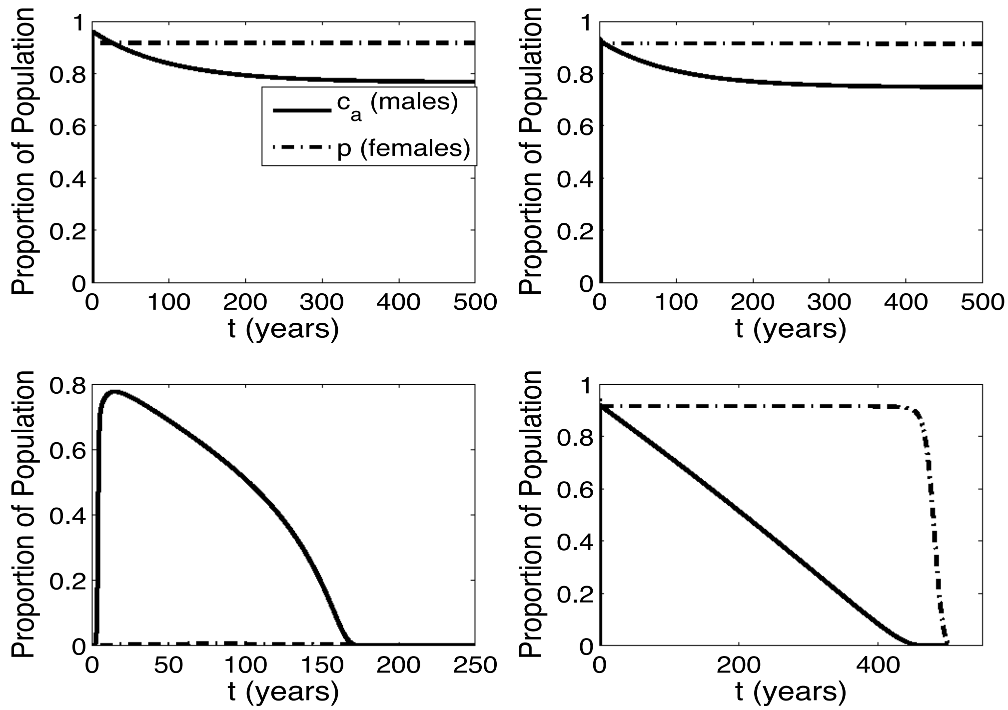

Figure 6: c_a and p as a coupled system. Upper left panel $\mathcal{R}_m > 1$, $\mathcal{R}_f > 1$; Upper right panel: $\mathcal{R}_m > 1$, $\mathcal{R}_f^c < \mathcal{R}_f < 1$; Lower left panel: $\mathcal{R}_f < \mathcal{R}_f^c$; Lower left panel: $\mathcal{R}_m < 1$,$\mathcal{R}_f > 1$.

model and display only the c_a and p proportions, as these are our target populations. In this figure the only state-dependent parameters are those coupling the two systems, that is the recruitment rates, β, ρ and η_1. In the upper left panel in Figure 6, we have used the standard parameter set. This best represents the current situation, with $\mathcal{R}_m > 1$ and $\mathcal{R}_f > 1$. Not surprisingly, both c_a and p reach a positive steady state. When $\mathcal{R}_f$ is decreased to the range $\mathcal{R}_f^c < \mathcal{R}_f < 1$ and $\mathcal{R}_m$ is still above 1, again both populations reach a positive steady state. This behavior is the same as was exhibited in the uncoupled models, and further exhibits that targeting the sex workers themselves with law enforcement does not reduce or even affect the scale of prostitution prevalence. While it is possible to eliminate prostitution by decreasing $\mathcal{R}_f$ to be below $\mathcal{R}_f^c$ alone (lower left panel of Figure 6, this is extremely unrealistic, and not possible in practice. If, however, $\mathcal{R}_m < 1$, and the male active customer population approaches the trivial equilibrium, the demand for prostitution is eliminated. We have shown previously that relatively modest increases in arrest rates for these men could produce this effect, thereby justifying a shift in efforts.

Thus, only through increased attention being paid to the male customers by law enforcement officials, could prostitution be effectively controlled. Furthermore, if the efforts are noticeable enough to impact the customer's habits without being arrested (i.e. by reducing δ_m and ω even slightly), the effects are amplified, and the need for direct enforcement is attenuated. We suggest that the focus of law enforcement programs focus on the driving force for this system, the male customer, who provides the demand for the sex worker industry. Without the customer, the system collapses.

7 Conclusion and Discussion

For the male model, holding all parameters constant we determined the active customer free equilibrium, C_aFE, and the basic reproductive number R_m. We proved local stability of the C_aFE and the existence of the active customer prevalent equilibrium, C_aPE. Furthermore we show that equilibrium exists for $R_m > 1$. For the female model, again holding all parameters constant we determined the prostitute free equilibrium, PFE, and the basic reproductive number R_f. We find the necessary conditions for backward bifurcation at $R_f = 1$, and show that there are many solutions to the system for which $R_m < 1$ approach the prostitute prevalent equilibrium, PPE. We find that reducing R_f is highly unlikely, suggesting that it is necessary to focus on the male system of demand in order to decrease the female sex worker population. We show that β is the key factor in decreasing R_f, and show that ω and γ_m are key factors in curbing the male demand. Furthermore, for the coupled model, when η_1 is treated as a function of δ_f, κ_f, and γ_f, we see that the female supply has little significance on the value of R_m, confirming the necessity to focus efforts on the male customer model in order to curb demand and therefore the sex worker populations. We also show that β, as a function of demand, appears to have a greater effect on the steady states of the female system, indicating the importance of decreasing the male demand in order to decrease β enough to change R_f. Through analysis we have shown that the male dynamics are not very sensitive to the changes in the female population. Furthermore, numerical solutions of the coupled model show that for $R_m < 1$, the the population of C_a approahes 0 and the male population approaches the C_aFE. Therefore, only via increased attention paid to the customers, related to law enforcement by officials, could the number of sex workers be effectively controlled.

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9 appendix

Local Stability of the C_aFE

To determine the stability of the C_aFE in a local neighborhood, we examine the eigenvalues of the Jacobian, or the linearized system, at the C_aFE.

$$\mathbf{J}|_{C_aFE} = \begin{pmatrix} -\eta_2 g^* - \mu_m & -\eta_2 c^* & -\eta_1 c^* + \omega & (1-\delta_m)\kappa_m \\ \eta_2 g^* & \eta_2 c^* - \mu & 0 & 0 \\ 0 & 0 & \eta_1 c^* - (\omega + \gamma_m + \mu_m) & \delta_m \kappa_m \\ 0 & 0 & \gamma_m & -(\mu_m + \kappa_m) \end{pmatrix}$$

The characteristic equation $det(\mathbf{J} - \lambda\mathbf{I}) = 0$ is a quartic polynomial which simplifies to

$$\begin{aligned} &\left(\lambda^2 - (\eta_2 c^* - \eta_2 g^* - 2\mu_m)\lambda - \eta_2^2 g^* c^* (\eta_2 g^* + \mu_m)(\eta_2 c^* - \mu_m)\right) \\ &\quad * \left(\lambda^2 - (\eta_1 c^* - (\omega + \gamma_m + 2\mu_m + \kappa_m))\lambda - (\eta_1 c^* - (\omega + \gamma_m + \mu_m))(\mu_m + \kappa_m) - \delta_m \gamma_m \kappa_m\right) = 0. \end{aligned}$$

We can obtain closed form expressions for all four eigenvalues using the quadratic formula and can see that they all have $\mathfrak{Re}\lambda < 0 \iff \mathcal{R}_m < 1$. We begin with the first quadratic polynomial from above.

$$\lambda_{1,2} = \frac{1}{2}(B \pm \sqrt{B^2 - 4C})$$

where $B = \eta_2(1 - 2g^*) - 2\mu_m$ using $g^* = 1 - c^*$.

$$B = \frac{\mu_m}{\eta_2} - \sqrt{(1 - \frac{\mu_m}{\eta_2})^2 + 4\frac{\mu_m}{\eta_2}(1 - \Lambda)} < \frac{\mu_m}{\eta_2} - \sqrt{(1 - \frac{\mu_m}{\eta_2})^2} = -1 < 0.$$

C can be rewritten as $C = -\eta_2^3 g^* c^* (\eta_2 g^* + \mu_m)(c^* - \frac{\mu}{\eta_2})$. The sign of C is clearly determined by $c^* - \frac{\mu}{\eta_2}$ since all parameters are positive, which we can determine by looking at this by substituting in for c^*.

$$c^* - \frac{\mu_m}{\eta_2} = \frac{1}{2}\left(1 - \frac{\mu_m}{\eta_2} - \sqrt{(1 - \frac{\mu_m}{\eta_2})^2 + 4\frac{\mu_m}{\eta_2}(1 - \Lambda)}\right) < \frac{1}{2}\left(1 - \frac{\mu_m}{\eta_2} - \sqrt{(1 - \frac{\mu_m}{\eta_2})^2}\right) = 0.$$

So $C > 0$ which implies $\sqrt{B^2 - 4C} < B$. Since $B < 0$, we have that $\lambda_1 < 0$ and $\lambda_2 < 0$. For $\lambda_{3,4}$ we turn to the second quadratic in the characteristic equation. Again we note that the eigenvalues have the form

$$\lambda_{3,4} = \frac{1}{2}(B \pm \sqrt{B^2 + 4C}))$$

and the signs of B and C will determine the sign of λ_3 and λ_4.

$$\begin{aligned}
B &= \eta_1 c^* - (\omega + \gamma_m + 2\mu_m + \kappa_m) \\
&= (\omega + \gamma_m + \mu_m)\left(\frac{\eta_1 c^*}{\omega + \gamma_m + \mu_m} - \frac{\mu_m + \kappa_m}{\omega + \gamma_m + \mu_m} - 1\right) \\
\text{Note } \mathcal{R}_m < 1 &\implies \frac{\eta_1 c^*}{\omega + \gamma_m + \mu_m} < 1 \\
&\implies B < 0
\end{aligned}$$

$$\begin{aligned}
C &= (\eta_1 c^* - (\omega + \gamma_m + \mu_m))(\mu_m + \kappa_m) + \delta_m \gamma_m \kappa_m \\
&= (\omega + \gamma_m + \mu_m)\left[\left(\frac{\eta_1 c^*}{\omega + \gamma_m + \mu_m} - 1\right)(\mu_m + \kappa_m) + \frac{\delta_m \gamma_m \kappa_m}{\omega + \gamma_m + \mu_m}\right] \\
&= (\omega + \gamma_m + \mu_m)(\mu_m + \kappa_m)\left[\frac{\eta_1 c^*}{\omega + \gamma_m + \mu_m} + \delta_m \frac{\gamma_m}{\omega + \gamma_m + \mu_m}\frac{\kappa_m}{\mu_m + \kappa_m} - 1\right] \\
&= (\omega + \gamma_m + \mu_m)(\mu_m + \kappa_m)(\mathcal{R}_m - 1) \\
&< 0 \iff \mathcal{R}_m < 1
\end{aligned}$$

$C < 0 \implies B > \sqrt{B^2 + 4C}$. Then $B < 0 \implies \lambda_{3,4} < 0$, and we have that $\mathfrak{Re}(\lambda_i) < 0 \iff \mathcal{R}_m < 1$ for $i = 1, 2, 3, 4$.