

A Dynamical Interpretation of the Three-Strikes Law

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Abstract

California's Three Strikes Law has been in effect since 1994. Advocates of this policy claim it acts as a deterrent for violent crime; yet critics allege it acts solely as an incapacitant—a device used to segregate a population of “undesirables” from the total population in an attempt to lower criminal susceptibility. To determine the true relationship between these two intimately connected phenomena, we construct a dynamical model of the Three-Strikes Law within the framework of inner-city communities located in Los Angeles County. We then compare this model to one of Los Angeles County before California implemented the Three-Strike policy—the classical incarceration model. Through qualitative analysis we determine the basic reproductive number, R_0 , for each of the models. Using numerical simulations, we then determine the net change in the total population of reformed inmates and the total number of incarcerated individuals due to the Three-Strikes Law. We also analyze the impact of population density on crime rates in states that utilize the Three-Strikes Law. Finally, we construct and examine a hypothetical One-Strike model to determine the impact of different strike policies on the reformed, criminal and incarcerated populations. We find that the Three-Strikes policy deters crime better than the classical incarceration policy in densely populated areas like Los Angeles County. In the context of population density, the Three-Strikes Law is a better deterrent in a sparsely populated region than a densely populated region. The optimal policy is found to be one that consists of more than three strikes.

1 Introduction

California’s Three-Strikes Law was instituted in 1994 in response to the rising crime rates and media sensationalism of crimes committed by repeat offenders [13, p. 31]. Though other states employ similar laws, California has by far the strictest Three-Strikes standards [10]. Under this policy, those individuals who are convicted of a second “striking” offence automatically receive double the sentence. When convicted of the third “striking” offence, an individual receives a sentence of life in prison [13, p. 56]. Specifically,

“three-strikers” are eligible for parole but must serve a minimum term that is the greater of (a) three times the usual sentence, (b) a minimum term of 25 years, or (c) another term stipulated by other sentencing statutes (e.g., life without parole or death). [13, p. 56]

According to the Legislative Analyst’s Office, California’s nonpartisan fiscal and policy advisor, the rationale for the Three-Strikes Law is twofold: sentence enhancements restrict the ability of repeat offenders to commit additional crimes by removing them from the population, and the threat of long time incarceration discourages some offenders from committing new crimes. Clearly, this line of thinking reflects the policy’s ability to both deter and incapacitate a population susceptible to criminal activity.

Legislative rhetoric about the Three-Strikes Law is extremely controversial [10]. In 2003, *Ewing vs. California* questioned the constitutionality of the Three-Strikes Law, stating that it violates the eighth amendment—prohibition against cruel and unusual punishment [6]. Furthermore, according to the Urban Institute’s study “Did Getting Tough on Crime Pay?”, nationwide reforms have caused a significant increase in prison populations. This study asserts that, “there has been a geographical clustering of incarceration...[and] the vast majority of persons admitted into prisons...have come from comparatively few large urban areas” [9].

In recent years, the violent crime rate has decreased; the Urban Institute claims that imprisonment is just one of the many reasons. However, this decrease has not come without its own costs. As of December 31, 2004, there were almost 43,000 inmates in California serving time under the Three-Strikes Law; of this population more than 35,000 are second strikers and 7,500 are third strikers [6]. The statistics indicate a bias toward members of urban communities. Due to the demographic make-up of urban populations, this law has significant implications for minorities. Although they make up only 7 percent of the total population, over 43 percent of the criminals serving their sentence under the Three-Strikes law are black [13, p. 113]. Ten years after the inception of the Three-Strikes policy, blacks and hispanics are still over-represented in prison populations [13, p. 114].

According to the California Department of Corrections and Rehabilitation (CDCR), 44 percent of all inmate strikers are convicted of a serious or violent offense, while 56 percent are convicted of nonviolent or nonserious offences [9]. Our model focuses on “striking” crimes that are termed violent. Examples of such violent offences, as defined under California State law (Penal Code 667.5), include murder, robbery, rape and other sex crimes [6].

Hence, violent offenders will be the main focus of our investigation. Moreover, due to the fact that we are dealing with a deterministic model, concentrating on violent offenders will eliminate any variation in parameters caused by judiciary discretion.

In our research, we model criminal activity as an infection targeting urban populations. Our main assumption is that new criminals are produced through contact with existing criminals. Using three models, we determine the relationship between deterrence and “incapacitation”: the first model represents the classical incarceration system, the second model represents incarceration under the Three-Strikes Law and the third is a hypothetical model in which a population is subject to a One-Strike policy. Through numerical simulation and comparative analysis, we determine whether the Three-Strike Policy is effective at deterring crime. Specifically, using numerical methods and data from Los Angeles County, we determine the total number of incarcerated and reformed individuals for each model. In addition, we analyze the consequences of population density on the efficacy of the Three-Strikes policy. Using published data from Montana and New Mexico, we determine the optimal population density for which the Three-Strikes policy is an effective deterrent. We define optimal population density as the density at which the proportion of reformed individuals to criminals, and the proportion of reformed individuals to incarcerated individuals, is maximized.

Unfortunately, there has been little investigation into the dynamics of the Three-Strikes Law [10]. Many sociological models exist, but they rely heavily on statistical and probabilistic methods. These investigations will no doubt be helpful in parameter estimation, but offer little in the way of guidance when it comes to dynamical analysis. This paper builds on the findings of two quantitative studies into the dynamics of urban communities: one study analyzes the relationship between poverty and crime, and the other specifically focuses on the Three-Strikes policy. The former, “The Dynamics of Poverty and Crime,” offers significant insight into the plight of the urban community, while the latter, “Fear of the First Strike,” serves as a relevant precursor to this paper.

The model used in “The Dynamics of of Poverty and Crime” is similar to our classical incarceration model (or in this investigation, the Infinite-Strikes model); it is a deterministic five dimensional system of ODEs with a constant population. In addition, the model described in Zhao et al. has an “at risk” class that can be influenced by individuals that are criminals. This paper is an excellent reference in terms of methods for determining the stability of our crime-free equilibrium. However, the focus of this investigation is the most cost-efficient way to lower criminality [14]. Clearly, the economics of incarceration is not our focus, but the results still offer insight into our research problem. It is interesting to note that in their model, the authors assume “the problem of crime is alleviated by either decreasing poverty or by increasing the severity of the ensuing punishment” [14]. The study concluded that it is necessary to control both of these parameters in order to obtain the most cost-effective strategy to combat crime [14].

“Fear of the First Strike” offers an intriguing mathematical representation of the Three-Strikes Law. This investigation uses a theoretical model to show that laws implementing a strike system should deter individuals contemplating their first crime [10]. We

use this result directly in our Three-Strikes model. However, the mathematics used in Shepherd’s economic interpretation of the Three-Strikes Law has little significance with regards to our model. Nevertheless, her most significant conclusion, that “changing from a system with no repeat-offender laws to a full two- and three-strikes system would be expected to increase deterrence” [10], gives us a concrete finding with which we can compare and contrast our results.

In Section 2, we provide an outline of our models. First, we give a brief explanation of the general framework of both the Infinite-Strikes model and the Three-Strikes model. Then we define each of the classes, state our assumptions, and describe the parameters governing our models. Next, we present our flow charts and enumerate each system of ODEs. In Section 3, we present our analytic computations; this section also contains the hypothetical One-Strike Model. In Section 4, we provide parameter estimations, and in Section 5 we perform numerical simulations. In the final sections, we present our results and state our conclusions. An Appendix is included at the end of this paper that offers additional analysis and computations.

2 Model Description

In our analysis, we will be considering two models: one with the Three-Strikes Law and one without. In each model we divide our total population into five base classes: a class of susceptibles (S), a class of at-risk individuals who have already participated in criminal activity (C), a class of prisoners (I), a class of released criminals (R) and a class of reformed criminals (V)—see Figures 1 and 2. In order to accommodate the usage and effect of the Three-Strikes Law, we design a three-tiered model in which each tier represents the consequences of a strike. By incorporating this tiered system into one of our models, we provide an analytical distinction between each deliverance of a strike. Included in each model is an absorbing state—the reformed class. The reformed class is used to assess the deterrent effect of the Three-Strikes Law. The incapacitant effect is then determined by the influx or outflux of imprisoned criminals as measured by the growth or decline in each of the incarcerated classes.

The general characteristics of each class remain the same in both models. Susceptibles are defined as individuals between the ages of 18 and 49 who live in Los Angeles County inner-city communities. In the Three-Strikes model, the C , I and R classes are divided into sub-classes. Criminals are differentiated by the total number of strikes, i , they have on their records, where $i = 0, 1$ or 2 . The total number of criminals is then represented by C , where $C = C_0 + C_1 + C_2$. Once detained and tried, these criminals are then sent to jail. The I_j classes then represent the imprisoned population, which consists of criminals who have received j strikes, where $j = 1, 2$ or 3 . After being jailed for a certain amount of time, these criminals are then released. The R_k classes are the classes of criminals released with k strikes, where $k = 1$ or 2 . Should they choose to reform, these individuals are then grouped into the V class, which is the class of reformed criminals.

Having defined each of our classes, the next step is to clarify the meaning behind each of our transitions and to explain the various rates involved in those transitions. First, let us consider the transitions in the Three-Strikes model since all of the transitions in the Infinite-Strikes model appear in the Three-Strikes model. From $S \rightarrow C_0$, we assume that the susceptible population becomes involved in criminal activity only by making contact with criminals (i.e. people involved in criminal activity). The same is true for the transitions $R_1 \rightarrow C_1$ and $R_2 \rightarrow C_2$. In both models, we are using mass action incidence; we assume that population density is a factor in each system. Individuals in each system act simultaneously and affect each other through various contact processes. For the transitions of the form $C_i \rightarrow I_{i+1}$, we assume that for those individuals involved in criminal activity (regardless of the number of strikes on their record), it is only a matter of time before they are caught and incarcerated. In our model, we ignore the discretionary aspects of the trial process. We assume that individuals who participate in criminal activity are eventually caught, tried and sent to prison with an additional strike. After serving the time they were sentenced, offenders with one or two strikes are released from prison. The transitions $I_1 \rightarrow R_1$ and $I_2 \rightarrow R_2$ represent this phenomenon. After being released, the criminals are faced with a choice either to reform or to continue committing crimes. This choice to reform is represented in the transitions $S \rightarrow V$, $R_1 \rightarrow V$ and $R_2 \rightarrow V$. In essence, susceptible individuals can decide to remove themselves from criminal activity for life, thus moving immediately from the S class to the V class. A decision to reform can also be made after being released from prison with one or two strikes. The deterrent effect of the Three-Strikes Law is incorporated into these transitions in the form of the parameter ϵ . Moreover, the rates at which these transitions are made depend on the number of people who have already received three strikes and have been sent to prison for life, i.e. the number of people in the I_3 class. An individual's decision to reform depends on how intimidated he or she is by the prospect of receiving three strikes: the more people in I_3 , the more this "intimidation factor" increases. In both models, we are assuming that individuals are born into the susceptible class at a constant entry rate and that they leave each class at a constant departure rate.

The meanings associated with the transitions for the Infinite-Strikes model and the One-Strike model follow directly from the explanations above. The parameters for our models are described in Table 1.

Parameters	Description
β_0	$\beta_0 C$ is the rate at which individuals become criminals
β_1	$\beta_1 R_1$ is the rate at which first strikers commit their second crime
β_2	$\beta_2 R_2$ is the rate at which second strikers commit their third crime
ω_0	rate at which susceptible individuals decide to remove themselves from criminal activity for life
ω_1	rate of reform for first strikers
ω_2	rate of reform for second strikers
ψ_1	rate of release from prison for first strikers
ψ_2	rate of release from prison for second strikers
ϵ_0	deterrent rate for the susceptible population
ϵ_1	deterrent rate for the population of released criminals with one strike
ϵ_2	deterrent rate for the population of released criminals with two strikes
ϕ	rate of a criminal being caught, convicted and sent to jail
δ	rate of being sent to jail for life after only receiving one or two strikes
μ	the average time that individuals spend in our system

Table 1: Parameter List

Having explained the purpose of each class and the various transitions from one class to another, we are now ready to introduce the flow charts for each model and their corresponding systems of equations. The first flow chart is of the Infinite-Strikes model (see Figure 1); the only absorbing state is the reformed class V .

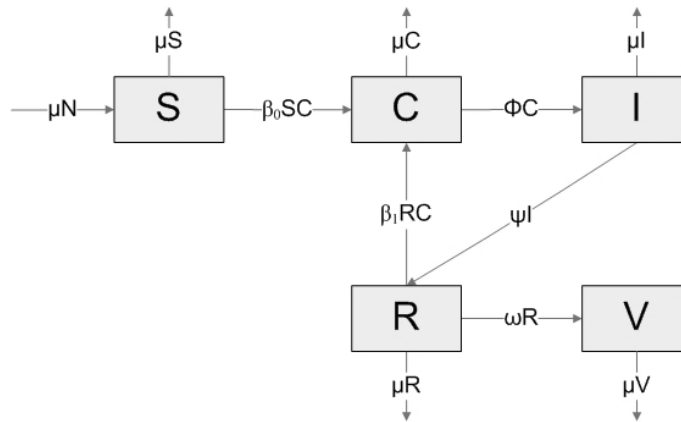


Figure 1: Flow Chart of an Incarceration System without a Three-Strikes Law

The Infinite-Strikes model is governed by the following system of ordinary differential equations:

$$\begin{aligned}
 \frac{dS}{dt} &= \mu N - \beta_0 CS - \mu S \\
 \frac{dC}{dt} &= \beta_0 CS + \beta_1 CR - \phi C - \mu C \\
 \frac{dI}{dt} &= \phi C - \psi I - \mu I \\
 \frac{dR}{dt} &= \psi I - \beta_1 CR - \omega R - \mu R \\
 \frac{dV}{dt} &= \omega R - \mu V \\
 N &= S + C + I + R + V
 \end{aligned}$$

The total population N is constant since $\frac{dN}{dt} = 0$. All parameter values are assumed to be positive.

Next, we have the flow chart for the Three-Strikes model, which expands the Infinite-Strikes model to incorporate the hierarchy of strikes present in the Three-Strikes Law. Notice, there are now two absorbing states, I_3 and V .

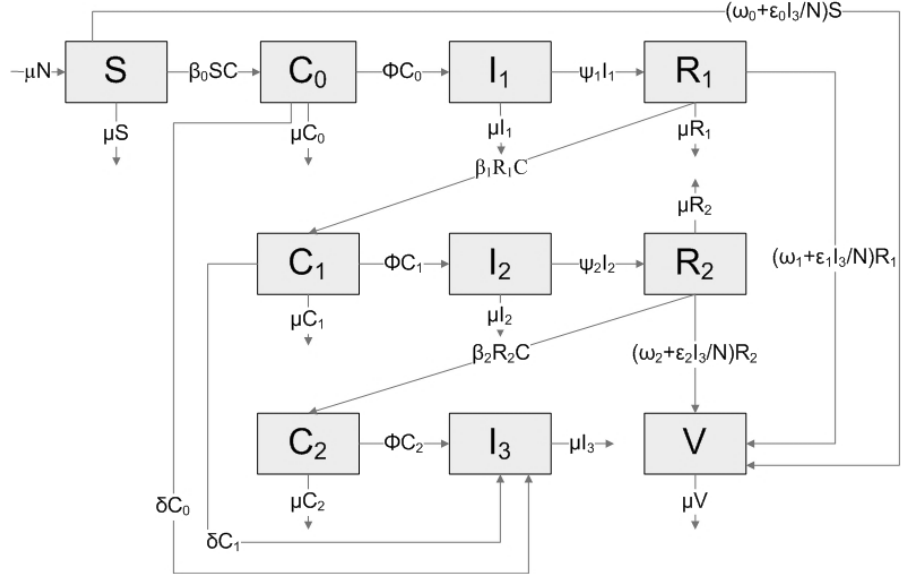


Figure 2: Flow Chart of an Incarceration System with a Three-Strikes Law

The Three-Strikes model is governed by the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \beta_0 CS - S \left(\omega_0 + \epsilon_0 \frac{I_3}{N} \right) - \mu S \\
\frac{dC_0}{dt} &= \beta_0 CS - \phi C_0 - \delta C_0 - \mu C_0 \\
\frac{dC_1}{dt} &= \beta_1 CR_1 - \phi C_1 - \delta C_1 - \mu C_1 \\
\frac{dC_2}{dt} &= \beta_2 CR_2 - \phi C_2 - \mu C_2 \\
\frac{dI_1}{dt} &= \phi C_0 - \psi_1 I_1 - \mu I_1 \\
\frac{dI_2}{dt} &= \phi C_1 - \psi_2 I_2 - \mu I_2 \\
\frac{dI_3}{dt} &= \phi C_2 + \delta C_0 + \delta C_1 - \mu I_3 \\
\frac{dR_1}{dt} &= \psi_1 I_1 - \beta_1 CR_1 - R_1 \left(\omega_1 + \epsilon_1 \frac{I_3}{N} \right) - \mu R_1 \\
\frac{dR_2}{dt} &= \psi_2 I_2 - \beta_2 CR_2 - R_2 \left(\omega_2 + \epsilon_2 \frac{I_3}{N} \right) - \mu R_2 \\
\frac{dV}{dt} &= S \left(\omega_0 + \epsilon_0 \frac{I_3}{N} \right) + R_1 \left(\omega_1 + \epsilon_1 \frac{I_3}{N} \right) + R_2 \left(\omega_2 + \epsilon_2 \frac{I_3}{N} \right) - \mu V \\
C &= C_0 + C_1 + C_2 \\
N &= S + C_0 + C_1 + C_2 + I_1 + I_2 + I_3 + R_1 + R_2 + V
\end{aligned}$$

Again, our total population is constant and all parameter values are assumed to be positive.

3 Analytic Computations

In this section, we analyze the various dynamical structures inherent in our two models. First, we calculate the basic reproductive number, R_0 , for each model. We then determine the necessary conditions to produce stable behavior in the crime-free equilibria of each model. The same analysis is done for the endemic equilibria, albeit in a different manner. After first determining whether a backward bifurcation is possible for the Infinite-Strikes model, we utilize the mechanics of bifurcation theory to categorize the emergence of endemic equilibria and classify their stability. We then transfer our findings into the framework of the Three-Strikes model and determine the appropriate endemic conditions for the Three-Strikes model.

3.1 Calculation of the Basic Reproductive Number, R_0 , for the Infinite-Strikes Model

To find the basic reproductive number for the Infinite-Strikes model, we use the Next-Generation Matrix method [12]. Since the total population $N = S + C + I + R + V$ is constant and V only appears in the equation for $\frac{dV}{dt}$, we can examine only the S, C, I and R classes.

$$\text{Let } \mathcal{F} = \begin{bmatrix} 0 \\ \beta_0 CS + \beta_1 CR \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} -\mu N + \beta_0 CS + \mu S \\ \phi C + \mu C \\ \phi C + \psi I + \mu I \\ \omega_1 R - \mu V \end{bmatrix}.$$

Considering only the infectious class, the C class, we formulate the Jacobian of \mathcal{F} and \mathcal{V} evaluated at the crime-free equilibrium $(N, 0, 0, 0)$:

$$F = \beta_0 N, \quad V = \phi + \mu$$

Taking the inverse of V , we have

$$V^{-1} = \frac{1}{\phi + \mu}$$

and so

$$FV^{-1} = \frac{\beta_0 N}{\phi + \mu}.$$

Hence, $R_0 = \frac{\beta_0 N}{\phi + \mu}$ for our Infinite-Strikes model.

Intuitively, this value of R_0 makes sense. We have the rate of transmission, $\beta_0 N$, multiplied by the average time per capita spent in the criminal class (the C class), $\frac{1}{\phi + \mu}$. The factor of N in R_0 incorporates our assumption of mass action incidence.

3.2 Calculation of the Basic Reproductive Number, R_0 , for the Three-Strikes Model

Similarly, we find the basic reproductive number for the Three-Strikes model using the Next-Generation Matrix method [12]. Due to the length of the 10×1 vectors \mathcal{F} and \mathcal{V} , we will not include them here. They are given in the Appendix.

Again, we consider only the infectious classes C_0 , C_1 and C_2 in the Jacobian of \mathcal{F} and \mathcal{V} . Evaluating these Jacobians at the crime-free equilibrium $(\frac{\mu N}{\omega_0 + \mu}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\omega_0 N}{\omega_0 + \mu})$ we have:

$$F = \begin{bmatrix} \frac{\beta_0 \mu N}{\omega_0 + \mu} & \frac{\beta_0 \mu N}{\omega_0 + \mu} & \frac{\beta_0 \mu N}{\omega_0 + \mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \phi + \delta + \mu & 0 & 0 \\ 0 & \phi + \delta + \mu & 0 \\ 0 & 0 & \phi + \mu \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} (\phi + \delta + \mu)^{-1} & 0 & 0 \\ 0 & (\phi + \delta + \mu)^{-1} & 0 \\ 0 & 0 & (\phi + \mu)^{-1} \end{bmatrix}$$

and so

$$FV^{-1} = \begin{bmatrix} \frac{\beta_0 \mu N}{(\omega_0 + \mu)(\phi + \delta + \mu)} & \frac{\beta_0 \mu N}{(\omega_0 + \mu)(\phi + \delta + \mu)} & \frac{\beta_0 \mu N}{(\omega_0 + \mu)(\phi + \mu)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has a dominant eigenvalue $\frac{\beta_0 \mu N}{(\omega_0 + \mu)(\phi + \delta + \mu)}$. Hence,

$$R_0 = \frac{\beta_0 \mu N}{(\omega_0 + \mu)(\phi + \delta + \mu)} = \left(\frac{\beta_0 N}{\phi + \delta + \mu} \right) \left(\frac{\mu}{\omega_0 + \mu} \right)$$

for our Three-Strikes model.

Once again, the R_0 that is calculated makes intuitive sense, especially once it is factored appropriately. In the first term, we have the rate of transmission, $\beta_0 N$, multiplied by the average time per capita spent in the C_0 class, $\frac{1}{\phi + \delta + \mu}$. The second term, $\frac{\mu}{\omega_0 + \mu}$, is the proportion of the population that is susceptible to criminal influence at the crime-free equilibrium. Again, the factor of N represents the fact that we are using mass action incidence in our model.

Notice that the R_0 for the Three-Strikes model reduces to the R_0 for the Infinite-Strikes model if we set $\delta = 0$ and $\omega_0 = 0$. Hence, if we remove the transitions in the Three-Strikes model involving δ and ω_0 , we essentially get a simplified version of the Infinite-Strikes model. Thus, the R_0 's calculated for both models are consistent with the flow charts.

3.3 Stability of the Crime-Free Equilibrium for the Infinite-Strikes Model

For our Infinite-Strikes model, we determined the crime-free equilibrium to be $(N, 0, 0, 0)$. In order to study the stability of this equilibrium, we first compute the Jacobian, J , of this system.

$$J = \begin{pmatrix} -\beta_0 C - \mu & -\beta_0 S & 0 & 0 \\ \beta_0 C & \beta_1 S + \beta_1 R - \phi - \mu & 0 & 0 \\ 0 & \phi & -\psi - \mu & 0 \\ 0 & -\beta_1 R & \psi & -\beta_1 C - \omega_1 - \mu \end{pmatrix}$$

Substituting our crime-free equilibrium into our Jacobian matrix, we end up with a matrix whose eigenvalues are simply its diagonal entries.

$$\begin{pmatrix} -\mu & -\beta_0 N & 0 & 0 \\ 0 & \beta_0 N - \phi - \mu & 0 & 0 \\ 0 & \phi & -\psi - \mu & 0 \\ 0 & 0 & \psi & -\omega_1 - \mu \end{pmatrix}$$

Therefore, $\lambda_1 = -\mu$, $\lambda_2 = \beta_0 N - \phi - \mu$, $\lambda_3 = -\psi - \mu$ and $\lambda_4 = -\omega_1 - \mu$. Our crime-free equilibrium is stable if all of these eigenvalues are negative. Since our parameters are all positive, we can automatically conclude that $\lambda_1 < 0$, $\lambda_3 < 0$ and $\lambda_4 < 0$. For λ_2 , we see that the condition $\phi + \mu > \beta_0 N$ must be met in order for it to be negative. Hence, the crime-free equilibrium $(N, 0, 0, 0)$ is stable if $\phi + \mu > \beta_0 N$ and unstable otherwise. This condition corresponds precisely to the R_0 for this model. Therefore, the crime-free equilibrium is stable if $R_0 < 1$ and unstable otherwise and we can state this result as the following lemma:

Lemma 1. *For the Infinite-Strikes model, if $R_0 < 1$, then the crime-free equilibrium is locally asymptotically stable.*

3.4 Stability of the Crime-Free Equilibrium for the Three-Strikes Model

For our Three-Strikes model, we found the crime-free equilibrium to be

$$\left(\frac{\mu N}{\omega_0 + \mu}, 0, 0, 0, 0, 0, 0, 0, \frac{\omega_0 N}{\omega_0 + \mu} \right).$$

Performing the usual stability analysis on this system of ODEs, we obtained the following eigenvalues of the Jacobian matrix evaluated at the crime-free equilibrium:

$$\begin{aligned}
\lambda_0 &= -\phi - \delta - \mu \\
\lambda_1 &= -\phi - \mu \\
\lambda_2 &= -\omega_0 - \mu \\
\lambda_3 &= \frac{\beta_0 \mu N}{\omega_0 + \mu} - \phi - \delta - \mu \\
\lambda_4 &= -\psi_1 - \mu \\
\lambda_5 &= -\psi_2 - \mu \\
\lambda_6 &= -\mu \\
\lambda_7 &= -\mu \\
\lambda_8 &= -\omega_1 - \mu \\
\lambda_9 &= -\omega_2 - \mu
\end{aligned}$$

All of the eigenvalues are clearly negative, except for λ_3 . In order for λ_3 to be negative, and consequently the crime-free equilibrium to be stable, the condition $(\phi + \delta + \mu)(\omega_0 + \mu) > \beta_0 \mu N$ must be met. Again, this condition corresponds directly to the R_0 found for this model. Thus, in terms of R_0 , the crime-free equilibrium is stable if $R_0 < 1$ and unstable otherwise. This result is stated in Lemma 2.

Lemma 2. *For the Three-Strikes model, if $R_0 < 1$, then the crime-free equilibrium is locally asymptotically stable.*

3.5 Endemic Equilibrium Analysis for the Infinite-Strikes Model

In determining the endemic equilibrium $(S^*, C^*, I^*, R^*, V^*)$ for the Infinite-Strikes Model, we first need to find S^* , I^* , R^* and V^* in terms of C^* . Doing so, we generate the following system of equations:

$$\begin{aligned}
S^* &= \frac{\mu N}{\beta_0 C^* + \mu} \\
I^* &= \frac{\phi C^*}{\psi + \mu} \\
R^* &= \frac{\psi \phi C^*}{(\psi + \mu)(\beta_1 C^* + \omega_1 + \mu)} \\
V^* &= \frac{\omega_1 \psi \phi C^*}{\mu(\psi + \mu)(\beta_1 C^* + \omega_1 + \mu)}
\end{aligned}$$

where C^* is given by the expression

$$\beta_0 C^* S^* + \beta_1 C^* R^* - \phi C^* - \mu C^* = 0.$$

Substituting S^* and R^* into the equation for C^* , we obtain the expression

$$\frac{\beta_0 \mu N C^*}{\beta_0 C^* + \mu} + \frac{\beta_1 \psi \phi (C^*)^2}{(\psi + \mu) (\beta_1 C^* + \omega_1 + \mu)} - \phi C^* - \mu C^* = 0.$$

Simplifying further, we have

$$\phi + \mu = \frac{\beta_0 \mu N}{\beta_0 C^* + \mu} + \frac{\beta_1 \psi \phi C^*}{(\psi + \mu) (\beta_1 C^* + \omega_1 + \mu)}.$$

Multiplying by the common denominator and dividing through by $\psi + \mu$, we arrive at the following expression in terms of C^* :

$$\beta_0 \mu (\beta_1 C^* + \omega_1 + \mu) N + \beta_1 \psi (\beta_0 C^* + \mu) \frac{\psi}{\psi + \mu} C^* - (\phi + \mu) (\beta_0 C^* + \mu) (\beta_1 C^* + \omega_1 + \mu) = 0.$$

Collecting like terms, we then have a quadratic equation in terms of C^* :

$$\begin{aligned} & [\beta_0 \beta_1 \left(\frac{\phi \psi}{\mu + \psi} - (\mu + \phi) \right)] (C^*)^2 + [\mu \beta_0 \beta_1 N + \beta_1 \phi \mu \frac{\psi}{\mu + \psi} - (\mu + \phi) (\beta_1 \mu + \beta_0 (\mu + \omega_1))] C^* \\ & + \mu \beta_0 N (\mu + \omega_1) - \mu (\mu + \phi) (\mu + \omega_1) = 0 \end{aligned} \quad (1)$$

Now let $x \equiv \frac{C^*}{N}$ and $f(x) = ax^2 + bx + c = 0$ where

$$a = (\beta_0 N)(\beta_1 N) \left(\frac{\phi \psi}{\mu + \psi} - (\mu + \phi) \right),$$

$$b = (\beta_0 N)(\beta_1 N) \mu + (\beta_1 N) \mu \left(\frac{\phi \psi}{\mu + \psi} \right) - (\beta_1 N) \mu (\mu + \phi) - (\beta_0 N) (\mu + \phi) (\mu + \omega_1),$$

$$c = (\beta_0 N) \mu (\mu + \omega_1) - \mu (\mu + \phi) (\mu + \omega_1).$$

We can rewrite a as

$$-(\beta_0 N)(\beta_1 N) \mu \left(1 + \frac{\phi}{\mu + \psi} \right)$$

since

$$\left(\frac{\phi \psi}{\mu + \psi} - (\mu + \phi) \right) = -\mu - \phi \left(1 - \frac{\psi}{\mu + \psi} \right) = -\mu - \phi \left(\frac{\mu}{\mu + \psi} \right) = -\mu \left(1 + \frac{\phi}{\mu + \psi} \right).$$

We can then pull out a factor of μ from each term of $f(x)$, i.e. from a , b and c . Letting $b_0 \equiv \beta_0 N$ and $b_1 \equiv \beta_1 N$, we have the following simplified expressions for a , b and c :

$$a = -b_0 b_1 \left(1 + \frac{\phi}{\mu + \psi} \right) \quad (2)$$

$$b = b_0 b_1 - b_0 \left(\frac{\mu + \phi}{\mu} \right) (\mu + \omega_1) - b_1 \mu \left(1 + \frac{\phi}{\mu + \psi} \right) \quad (3)$$

$$c = (b_0 - (\mu + \phi))(\mu + \omega_1)$$

Now we can obtain solutions in the interval $[0,1]$ to $f(x) = 0$. In other words, we can find the endemic equilibria of our system given the various relationships between the parameters. First, let us further simplify the equation for c by incorporating the basic reproductive number R_0 into the expression. Since $R_0 = \frac{\beta_0 N}{\mu + \phi} = \frac{b_0}{\mu + \phi}$ for this model, we can rewrite c as

$$c = (\mu + \phi) (\mu + \omega_1) (R_0 - 1). \quad (4)$$

We see that $R_0 > 1 \Leftrightarrow c > 0$ and $R_0 < 1 \Leftrightarrow c < 0$. Hence, the existence of our endemic equilibria depends on the various conditions for our parameters. Specifically, we have the following set of conditions, which is identical to the one found in [8]:

- (i) If $R_0 > 1 \Leftrightarrow c > 0$, then there is only one endemic equilibrium.
- (ii) If $R_0 < 1 \Leftrightarrow c < 0$, $b > 0$ and $b^2 - 4ac > 0$, then there are two endemic equilibria.
- (iii) If $b^2 - 4ac < 0$, then there is no endemic equilibrium. (This implies $R_0 < 1$.)

The first item requires no further elaboration in terms of parameter conditions. The fact that $c = f(0) > 0$ and $a < 0$ is sufficient in guaranteeing precisely one endemic equilibrium. (Note that $f(1) < 0$. If $f(1) \geq 1$, then the solutions we obtain are meaningless.)

The second item needs simplification in terms of the conditions $b > 0$ and $b^2 - 4ac > 0$. Let us consider the first condition $b > 0$. This condition is necessary because $a < 0$ and we want the x-coordinate of the vertex of $f(x)$ to be positive, i.e. we want $\frac{-b}{2a} > 0$. Essentially, this condition can be reduced to two different sets of parameter conditions, both of which are equally valid in the study of the endemic equilibria. We will consider only one of the sets in this section. The other will be explained in the Appendix.

First, let us factor out b_1 from the expression for b . Notice that one could have very well factored out a term of b_0 . This distinction in factorization is the reason why the condition $b > 0$ can be reduced to two sets of parameter conditions. Thus, we have $b = b_0 b_1 - b_0 \left(\frac{\mu + \phi}{\mu} \right) (\mu + \omega_1) - b_1 \mu \left(1 + \frac{\phi}{\mu + \psi} \right) = b_1 (b_0 - \mu \left(1 + \frac{\phi}{\mu + \psi} \right)) - b_0 (\mu + \phi) \left(\frac{\mu + \omega_1}{\mu} \right)$. We see that in order for $b > 0$, we first must have $b_0 > \mu \left(1 + \frac{\phi}{\mu + \psi} \right)$. Combining the R_0 condition gathered from the analysis on c and the b_0 condition just gotten, we then have the following criterion:

$$\mu \left(1 + \frac{\phi}{\mu + \psi} \right) < b_0 < \mu + \phi$$

or

$$\frac{\mu + \phi \left(\frac{\mu}{\mu + \psi} \right)}{\mu + \phi} < R_0 < 1.$$

Simplifying further we have

$$1 - \left(\frac{\phi}{\mu + \phi} \right) \left(\frac{\psi}{\mu + \psi} \right) < R_0 < 1.$$

Let us denote

$$K \equiv 1 - \left(\frac{\phi}{\mu + \phi} \right) \left(\frac{\psi}{\mu + \psi} \right) = \frac{\mu + \phi \left(\frac{\mu}{\mu + \psi} \right)}{\mu + \phi} \quad (5)$$

so then

$$K < R_0 < 1.$$

In addition, we need $b_1(b_0 - \mu(1 + \frac{\phi}{\mu + \psi})) > b_0(\mu + \phi) \left(\frac{\mu + \omega_1}{\mu} \right) \Rightarrow \frac{b_1}{\mu + \phi} > \left(\frac{b_0}{b_0 - (\mu + \phi) \left(\frac{\mu}{\mu + \psi} \right)} \right) \left(\frac{\mu + \omega_1}{\mu} \right)$.

Letting $R_1 \equiv \left(\frac{b_1}{\mu + \phi} \right) \left(\frac{\mu}{\mu + \omega_1} \right)$, we then have the condition $R_1 > \left(\frac{b_0}{b_0 - (\mu + \phi) \left(\frac{\mu}{\mu + \psi} \right)} \right)$. Rewriting this inequality in terms of R_0 and K , we have as our second condition that $R_1 > \frac{R_0}{R_0 - K} > 1$. R_1 is essentially the reinfection number, i.e. the average number of released prisoners who are influenced into committing another crime by a typical criminal.

The condition $b^2 - 4ac > 0$ requires more work to determine its parameter conditions.

Let us first multiply $b^2 - 4ac$ by the expression $\frac{\left(\frac{\mu}{\mu + \omega_1} \right)^2}{(\mu + \phi)^4}$ where a , b and c are as specified in (2), (3) and (4). We can then analyze when $\frac{\left(\frac{\mu}{\mu + \omega_1} \right)^2}{(\mu + \phi)^4} (b^2 - 4ac) > 0$ which is equivalent to analyzing when $b^2 - 4ac > 0$. Distributing $\frac{\left(\frac{\mu + \omega_1}{\mu + \phi} \right)^2}{(\mu + \phi)^4}$, we first have

$$\left(\frac{\left(\frac{\mu}{\mu + \omega_1} \right)^2}{(\mu + \phi)^4} \right) b^2 = \left(\frac{\left(\frac{\mu}{\mu + \omega_1} \right)}{(\mu + \phi)^2} b \right)^2.$$

Substituting the expression for b and reducing, we have

$$\left(\frac{b_0 b_1 \mu}{(\mu + \omega_1)(\mu + \phi)^2} - \frac{b_0}{\mu + \phi} - \frac{b_1 \mu^2}{(\mu + \omega_1)(\mu + \phi)^2} - \frac{b_1 \mu^2 \phi}{(\mu + \omega_1)(\mu + \psi)(\mu + \phi)^2} \right)^2. \quad (6)$$

Recall that $R_0 = \frac{b_0}{\mu + \phi}$ and $R_1 = \left(\frac{b_1}{\mu + \phi} \right) \left(\frac{\mu}{\mu + \omega_1} \right)$. Hence, we can reduce (6) to the following:

$$\left(R_0 R_1 - R_0 - R_1 \left(\frac{\mu}{\mu + \phi} + \frac{\mu \phi}{(\mu + \phi)(\mu + \psi)} \right) \right)^2.$$

Letting $k = \frac{\mu}{\mu + \psi}$ and rewriting (5) as $K = \frac{\mu + k\phi}{\mu + \phi}$, we then have

$$(R_0 R_1 - R_0 - R_1 K)^2.$$

Now consider

$$-4 \left(\frac{\mu}{(\mu + \omega_1)(\mu + \phi)^2} \right)^2 ac.$$

Substituting the expressions for a and c , we have

$$\left(4b_0b_1 \left(\frac{\mu}{(\mu + \omega_1)(\mu + \phi)^2} \right)^2 + \frac{4b_0b_1\phi}{\mu + \psi} \left(\frac{\mu}{(\mu + \omega_1)(\mu + \phi)^2} \right)^2 \right) ((b_0 - (\mu + \phi))(\mu + \omega_1))$$

which reduces to

$$4R_0R_1 \left((R_0 - 1) \left(\frac{\mu}{\mu + \phi} + \frac{\phi\mu}{(\mu + \phi)(\mu + \psi)} \right) \right) = 4R_0R_1(R_0 - 1)K.$$

Hence,

$$\frac{\left(\frac{\mu}{\mu + \omega_1} \right)^2}{(\mu + \phi)^4} (b^2 - 4ac) = (R_0R_1 - R_0 - R_1K)^2 + 4R_0R_1(R_0 - 1)K.$$

Simplifying and grouping like terms of R_1 , we obtain the following quadratic form in terms of R_1 :

$$((R_0 - K)^2) R_1^2 - (2R_0(R_0(2K - 1)) - K) R_1 + R_0^2 \equiv g(R_1)$$

which we require to be positive. Consider the case when $g(R_1) = 0$. Then

$$R_1 = \frac{-R_0(R_0(2K - 1) - K)}{(R_0 - K)^2} \pm \frac{\sqrt{(-2R_0(R_0(2K - 1) - K))^2 - 4(R_0 - K)^2R_0^2}}{2(R_0 - K)^2}.$$

This simplifies to

$$R_1 = \frac{-R_0}{(R_0 - K)^2} \left(2KR_0 - (R_0 + K) \pm 2\sqrt{R_0K(K - 1)(R_0 - 1)} \right).$$

Let

$$R_1^+ = \frac{-R_0}{(R_0 - K)^2} \left(2KR_0 - (R_0 + K) - 2\sqrt{R_0K(K - 1)(R_0 - 1)} \right)$$

$$R_1^- = \frac{-R_0}{(R_0 - K)^2} \left(2KR_0 - (R_0 + K) + 2\sqrt{R_0K(K - 1)(R_0 - 1)} \right).$$

Therefore, in order for $g(R_1) > 0$, we have either

- (i) $0 < R_1 < R_1^-$
- (ii) $R_1 > R_1^+$

However, we can reduce the number of conditions for R_1 down to simply one. In order to do this, we need to show the following claim.

Claim 1. *If $b > 0$, then $R_1^+ > \frac{R_0}{R_0 - K} > R_1^-$.*

Proof. Let us start with the first inequality. Rewrite R_1^+ as follows:

$$R_1^+ = \frac{R_0}{(R_0 - K)^2} \left(2\sqrt{R_0 K (K - 1)(R_0 - 1)} + (R_0 + K) - 2KR_0 \right).$$

We only need to show that

$$\frac{\left(2\sqrt{R_0 K (K - 1)(R_0 - 1)} + (R_0 + K) - 2KR_0 \right)}{(R_0 - K)} > 1,$$

which reduces to

$$\sqrt{R_0 K (K - 1)(R_0 - 1)} > KR_0 - 1.$$

Since $K < R_0 < 1$, $KR_0 - 1 < 0$ and $\sqrt{R_0 K (K - 1)(R_0 - 1)} > 0 \Rightarrow \sqrt{R_0 K (K - 1)(R_0 - 1)} > 0 > KR_0 - 1$. Hence,

$$R_1^+ = \frac{R_0}{(R_0 - K)} \left(\frac{2\sqrt{R_0 K (K - 1)(R_0 - 1)} + (R_0 + K) - 2KR_0}{(R_0 - K)} \right) > \frac{R_0}{R_0 - K}.$$

For the second inequality, let us first rewrite R_1^- as

$$R_1^- = \frac{R_0}{R_0 - K} \left(1 - \frac{2KR_0}{R_0 - K} - \frac{2\sqrt{R_0 K (K - 1)(R_0 - 1)}}{R_0 - K} \right).$$

Now consider the term

$$\begin{aligned} \left(1 - \frac{2KR_0}{R_0 - K} - \frac{2\sqrt{R_0 K (K - 1)(R_0 - 1)}}{R_0 - K} \right) &= 1 - \frac{2}{R_0 - K} (KR_0 + \sqrt{KR_0(K - 1)(R_0 - 1)}) \\ &< 1 - \frac{2KR_0}{R_0 - K} \\ &= 1 - \left(\frac{2}{\frac{1}{K} - \frac{1}{R_0}} \right) \\ &< 1 \end{aligned}$$

since $\frac{2}{\frac{1}{K} - \frac{1}{R_0}} > 0$. Hence, $R_1^- < \frac{R_0}{R_0 - K}$. □

Therefore, we can simply say that if $b > 0$, then $b^2 - 4ac > 0 \Leftrightarrow R_1 > R_1^+$. Our set of conditions can then be summarized as the following theorem.

Theorem 1a. *For our Infinite-Strikes model,*

(i) *If $R_0 > 1$, then there is only one endemic equilibrium.*

(ii) If $K < R_0 < 1$ and $R_1 > R_1^+$, then there are two endemic equilibria, where $K = 1 - \left(\frac{\phi}{\mu + \phi}\right) \left(\frac{\psi}{\mu + \psi}\right)$ and $R_1^+ = \frac{-R_0}{(R_0 - K)^2} \left(2KR_0 - (R_0 + K) - 2\sqrt{R_0K(K-1)(R_0-1)}\right)$. Otherwise, there is none.

Having determined the conditions for the emergence of our endemic equilibria, we are now interested in analyzing the stability of the endemic equilibria via the mechanics of bifurcation theory. From the conditions in Lemma 1 and the stability analysis of the crime-free equilibrium, we see that our system exhibits a backward bifurcation at the bifurcation point $R_0 = 1$ or when $\beta_0 = \frac{\mu + \phi}{N}$. Utilizing the parameter values estimated in section 4, we can then construct a bifurcation diagram reflecting the behavior of the endemic equilibria as R_0 varies (see Figure 3).

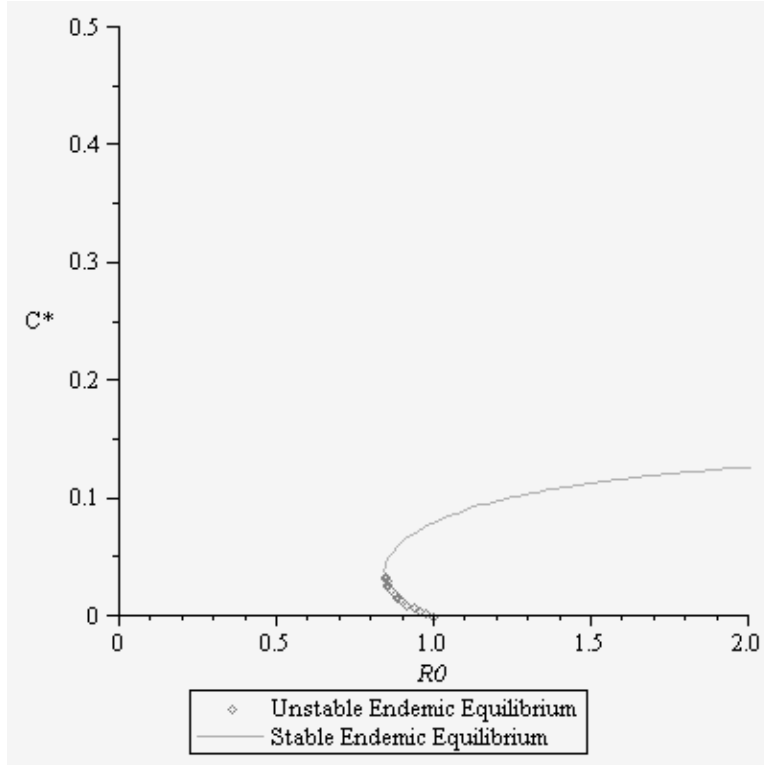


Figure 3: Backwards Bifurcation Diagram

The saddle-node bifurcation occurs at a certain point $0 < R_0^c < 1$. This point is determined by setting the discriminant $b^2 - 4ac$ of (1) equal to 0. We can write $b^2 - 4ac = 0$ as a quadratic equation in terms of R_0 with the coefficients in terms of R_1 :

$$((R_1 - 1)^2 + 4KR_1)R_0^2 - (2KR_1(R_1 + 1))R_0 + R_1^2K^2 = 0.$$

Solving for R_0 , we get

$$R_0 = \frac{R_1 + 1 \pm 2\sqrt{R_1(1-K)}}{\frac{(R_1-1)^2}{R_1K} + 4}.$$

Again, let

$$R_0^+ = \frac{R_1 + 1 + 2\sqrt{R_1(1-K)}}{\frac{(R_1-1)^2}{R_1K} + 4}$$

$$R_0^- = \frac{R_1 + 1 - 2\sqrt{R_1(1-K)}}{\frac{(R_1-1)^2}{R_1K} + 4}.$$

Then $R_0^c = R_0^+$, since we are interested in the larger of the roots. With the parameters estimated in section 4, R_0^c comes out to be approximately 0.84. Knowing R_0^c , we can refine the condition on R_0 even further in Lemma 1. We just need to verify that $R_0^+ > K$. We have $R_0^+ > K \Leftrightarrow$

$$R_1 + 1 + 2\sqrt{R_1(1-K)} > \frac{(R_1-1)^2}{R_1} + 4K$$

$$R_1^2 + R_1 + 2R_1\sqrt{R_1(1-K)} > (R_1-1)^2 + 4KR_1$$

$$2R_1\sqrt{R_1(1-K)} > -3R_1 + 1 + 4KR_1$$

$$2\sqrt{R_1(1-K)} > -3 + \frac{1}{R_1} + 4K$$

$$4R_1(1-K) > (4K + \frac{1}{R_1} - 3)^2$$

$$4(1-K)R_1^3 - (4K-3)^2R_1^2 - 2(4K-3)R_1 - 1 > 0$$

Let $h(R_1) \equiv 4(1-K)R_1^3 - (4K-3)^2R_1^2 - 2(4K-3)R_1 - 1$ and set $h(R_1) = 0$. The roots of h are $R_1^a = \frac{1}{4(1-K)}$, $R_1^b = 1 - 2K + 2\sqrt{-K(1-K)}$ and $R_1^c = 1 - 2K - 2\sqrt{-K(1-K)}$. Since $K < 1$ by definition, R_1^b and R_1^c are imaginary, so we only consider R_1^a . We then have two cases to analyze:

- (a) $R_0^+ > K \Leftrightarrow h(R_1) > 0 \Leftrightarrow 4K + \frac{1}{R_1} - 3 < 0 < 2\sqrt{R_1(1-K)}$
- (b) $R_0^+ > K \Leftrightarrow h(R_1) > 0 \Leftrightarrow 0 < 4K + \frac{1}{R_1} - 3 < 2\sqrt{R_1(1-K)}$

In the first case, we see that $h(R_1) > 0 \Leftrightarrow R_1 > R_1^a = \frac{1}{4(1-K)}$. This inequality is satisfied if $K < \frac{3}{4}$ since $K < \frac{3}{4} \Leftrightarrow \frac{1}{4(1-K)} < 1$ so hence $R_1 > 1 > \frac{1}{4(1-K)}$. In the second case, we see that $4K + \frac{1}{R_1} - 3 > 0 \Leftrightarrow \frac{1}{R_1} > 3 - 4K$ which is true if and only if $K \geq \frac{3}{4}$. Hence, the condition $R_0^+ > K$ is satisfied for all possible values of K where $0 < K < 1$. Theorem 1 can then be rewritten as follows:

Theorem 1b. *For our Infinite-Strikes model,*

(i) If $R_0 > 1$, then there is only one endemic equilibrium.

(ii) If $R_0^c < R_0 < 1$ and $R_1 > 1$, then there are two endemic equilibria, where $R_0^c = R_0^+ = \frac{R_1 + 1 + 2\sqrt{R_1(1-K)}}{\frac{(R_1-1)^2}{R_1K} + 4}$ and $R_1^+ = \frac{-R_0}{(R_0-K)^2} \left(2KR_0 - (R_0 + K) - 2\sqrt{R_0K(K-1)(R_0-1)} \right)$.
 Otherwise, there is none.

The fact that we have a backwards bifurcation has a significant meaning within our sociological framework. We see that even when $R_0 < 1$, endemic equilibria can still arise, specifically two endemic equilibria. In other words, even when the average number of susceptibles influenced into committing their first crime by a typical criminal is less than one, criminality can still persist. This is due to the fact that our reinfection number R_1 is greater than one, i.e., the average number of released prisoners who commit their next crime due to criminal influences is greater than one. Essentially, when $R_0 < 1$, our crime-free equilibrium is locally stable. There are no criminals in our population and no new criminals are produced in our system. At the bifurcation point $R_0 = 1$, the crime-free equilibrium becomes unstable and two endemic equilibria are born, one stable and one unstable. Normally in a forward bifurcation, the infection will have taken hold at this point, and increasing R_0 past one will have propagated the infection. However in a backwards bifurcation, there is an added effect induced by the presence of R_1 in our system. Increasing R_1 past one actually changes the direction of the bifurcation and consequently produces a saddle-node bifurcation point at R_0^c . Increasing R_0 past one simply makes the solution jump straight up to the unique stable endemic equilibrium. Thus, in the interval $R_0^c < R_0 < 1$, we see the impact of the released prisoners on the endemic equilibria, for they perpetuate the infection through their new crimes, even though the effect of the susceptible population's first crimes is subdued due to R_0 being less than one. Once R_0 is increased past one, there is only one endemic equilibrium and it is stable. Criminality then spreads indefinitely throughout the susceptible population.

As with most backwards bifurcations, the one in our model exhibits hysteresis [11]. As R_0 decreases past the bifurcation point via the stable endemic curve, we have to decrease R_0 even further in order to make the graph jump back to the bifurcation point [11]. In particular, we have to decrease R_0 down to R_0^c in order for the stable endemic equilibrium to become a stable crime-free equilibrium once again. Hence, there is a lack of reversibility as R_0 is varied in this system [11]. In a sociological context, this hysteresis phenomenon implies a sort of persistence in criminality. Even when the average number of susceptibles influenced into committing their first crime by criminals is less than one, criminality still exists. As mentioned before, this is due to the effect of R_1 on our system. It is not until this average number reaches a critical value, namely R_0^c , that criminality vanishes. Therefore, we see that criminality can easily pervade an urban population; we simply need to increase R_0 past one. However, it is much harder to rid that population of criminality once it takes hold, for it is not a simple matter of decreasing R_0 below one; R_0 has to be decreased even further in order for criminality to be eliminated.

3.6 Endemic Equilibrium Analysis for the Three-Strikes Model

Having established criteria for the number and stability of endemic equilibria for the Infinite-Strikes model, we hope to extend the analysis to the dynamics of the Three-Strikes model. Due to the complexity of the Three-Strikes model, we will resort to numerical analysis to determine most of our results. In particular, a modified version of Theorem 1 can be useful in our investigation of the Three-Strikes model. We see that the Three-Strikes model is immersed within the structure of the Infinite-Strikes model, so we can infer from our analysis on the Infinite-Strikes model that the Three-Strikes model will also exhibit a backwards bifurcation at the bifurcation point $R_0 = 1$. However, in this case, our endemic equilibria will be based on three infectious states, C_0 , C_1 and C_2 , or C where $C = C_0 + C_1 + C_2$. Our goal is to construct a similar bifurcation diagram for the Three-Strikes model.

First, we would like to determine the equations governing the infectious states as before with the Infinite-Strikes model. Solving for the endemic equilibrium, we arrive at the following set of quadratic equations for the infectious classes C_0 , C_1 and C_2 :

$$x(C_0) \equiv [(\phi + \delta + \mu)(\beta_0 + \frac{\epsilon_0 \delta}{\mu N})C_0^2 + [\beta_0 \mu N - (\phi + \delta + \mu)(\beta_0 C_1 + \beta_0 C_2 + \omega_0 + \frac{\epsilon_0 \mu C_2}{\mu N} + \frac{\epsilon_0 \delta C_1}{\mu N} + \mu)]C_0 + (\beta_0 \mu N)(C_1 + C_2) = 0 \quad (7)$$

$$y(C_1) \equiv -[(\phi + \delta + \mu)(\psi_1 + \mu)(\beta_1 + \frac{\epsilon_1 \delta}{\mu N})]C_1^2 + [\beta_1 \psi_1 \phi C_0 - (\phi + \delta + \mu)(\psi_1 + \mu)(\beta_1 C_0 + \beta_1 C_2 + \omega_1 + \frac{\epsilon_1 (\phi C_2 + \delta C_0)}{\mu N} + \mu)]C_1 + \beta_1 \psi_1 \phi C_0(C_0 + C_2) = 0 \quad (8)$$

$$z(C_2) \equiv -[(\phi + \mu)(\psi_2 + \mu)\beta_2 + \frac{\epsilon_2 \phi}{\mu N}]C_2^2 + [\beta_2 \psi_2 \phi C_1 - (\phi + \mu)(\psi_2 + \mu)(\beta_2 C_0 + \beta_2 C_1 + \omega_2 + \frac{\epsilon_2 (\delta C_0 + \delta C_1)}{\mu N} + \mu)]C_2 + \beta_2 \psi_2 \phi C_1(C_1 + C_0) = 0 \quad (9)$$

The mathematical complexity of these equations clearly poses a problem in our analysis of the endemic equilibria. Even when employing the use of various symbolic and numerical manipulators, we were not able to find solutions to this system. Hence, a graphical representation of this backwards bifurcation is not feasible at this point. However, we can utilize various graphical methods to verify that indeed we do have a backwards bifurcation for the Three-Strikes model. Through the process of trial and error, we find the saddle-node bifurcation point of the Three-Strikes backwards bifurcation to occur at approximately $R_0 = 0.508$. The derivation of this number is explained in section 5. From Lemma 2, we know that we have a stable crime-free equilibrium for $R_0 < 1$, which then becomes unstable as R_0 is increased past one. At the bifurcation point $R_0 = 1$, two endemic equilibria are created. As before with the backwards bifurcation of the Infinite-Strikes model, increasing R_1 past one will cause the bifurcation to change direction, thereby creating a saddle-node bifurcation point at $R_0 = 0.508$. As R_0 is increased past one, the solution jumps straight to the unique stable endemic equilibrium.

3.7 One-Strike Model

In this last part of our analytic section, we introduce a hypothetical incarceration model involving only one strike. In this model, an individual is imprisoned for life if he commits one violent offence. The purpose of this One-Strike model is to allow for analysis concerning the optimal number of strikes given to criminals. We define optimal to mean the most number of people to reform as a result of the various strike policies. We will inquire further into this issue in section 5. The transitional assumptions made for this model are equivalent to those for the Three-Strikes Model. Notice that once an individual is sent to I , the only way for him to leave I is by aging, i.e. in this model I is equivalent to I_3 in the Three-Strikes model. The flow chart for the One-Strike model is given in Figure 4.

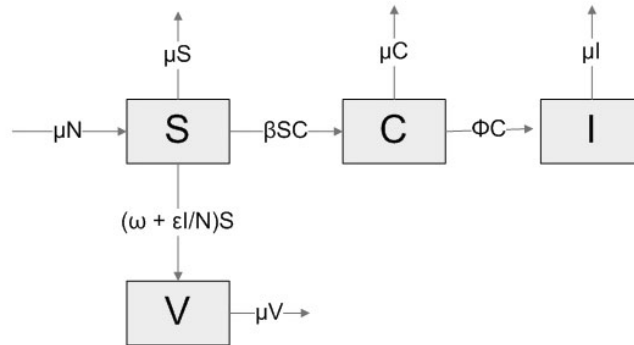


Figure 4: Hypothetical Model of an Incarceration System with a One-Strike Law

The One-Strike model is governed by the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \mu S - \beta CS - \left(\omega + \epsilon \frac{I}{N}\right) S \\
\frac{dC}{dt} &= \beta CS - (\mu + \phi)C \\
\frac{dI}{dt} &= \phi C - \mu I \\
\frac{dV}{dt} &= \left(\omega + \epsilon \frac{I}{N}\right) S - \mu V \\
N &= S + C + I + V
\end{aligned}$$

Again, the total population is constant and all parameter values are positive. We are also assuming mass action incidence in this model.

3.7.1 Calculation of the Basic Reproductive Number, R_0 , for the One-Strike Model

Once again, we will utilize the Next-Generation Matrix method to determine the R_0 for this model [12]. Let

$$\mathcal{F} = \begin{bmatrix} 0 \\ \beta SC \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} -\mu N + \beta SC + \mu S + \left(\omega + \frac{\epsilon I}{N}\right) S \\ (\mu + \phi) C \\ -\phi C + \mu I \\ -\left(\omega + \frac{\epsilon I}{N}\right) S + \mu V \end{bmatrix}.$$

Like in the previous two models, we consider only the infectious class C when calculating the Jacobian. Therefore, at the crime-free equilibrium $\left(\frac{\mu N}{\mu + \omega}, 0, 0, \frac{\omega N}{\mu + \omega}\right)$,

$$F = \frac{\beta \mu N}{\mu + \omega}, \quad V = \mu + \phi.$$

Taking the inverse of V , we have

$$V^{-1} = \frac{1}{\mu + \phi}$$

and finally

$$FV^{-1} = \frac{\beta \mu N}{(\mu + \phi)(\mu + \omega)}.$$

Thus, $R_0 = \frac{\beta \mu N}{(\mu + \phi)(\mu + \omega)} = \left(\frac{\beta N}{\mu + \phi}\right)\left(\frac{\mu}{\mu + \omega}\right)$. Again, we have the rate of transmission, βN , multiplied by the average time per capita spent in the C class, $\frac{1}{\mu + \phi}$. As with the R_0 in the Three-Strikes model, the term $\frac{\mu}{\mu + \omega}$ is the probability of an individual not reforming and remaining in the susceptible class, and the factor of N accounts for our mass action incidence assumption.

3.7.2 Stability of the Crime-Free Equilibrium for the One-Strike Model

For our hypothetical One-Strike model, we have a crime-free equilibrium at

$$\left(\frac{\mu N}{\mu + \omega}, 0, 0, \frac{\omega N}{\mu + \omega}\right).$$

In order to study the stability of this equilibrium, we first compute the Jacobian, J , of this system.

$$J = \begin{pmatrix} -\mu - \beta C - \omega - \frac{\epsilon I}{N} & -\beta S & -\frac{\epsilon S}{N} & 0 \\ \beta C & \beta S - \mu - \phi & 0 & 0 \\ 0 & \phi & -\mu & 0 \\ \omega + \frac{\epsilon I}{N} & 0 & \frac{\epsilon S}{N} & -\mu \end{pmatrix}$$

Substituting the crime-free equilibrium into our Jacobian matrix, we solve the characteristic equation $\det(J_{cfe} - \lambda I)$ for λ where

$$J_{cfe} = \begin{pmatrix} -\mu - \omega & \frac{\beta \mu N}{\mu + \omega} & -\frac{\epsilon \mu}{\mu + \omega} & 0 \\ 0 & \frac{\beta \mu N}{\mu + \omega} - \mu - \phi & 0 & 0 \\ 0 & \phi & -\mu & 0 \\ \omega & 0 & \frac{\beta \mu}{\mu + \omega} & -\mu \end{pmatrix}$$

Therefore, $\lambda_1 = -\mu - \omega$, $\lambda_2 = \frac{\beta \mu N}{\mu + \omega} - \mu - \phi$ and $\lambda_{3,4} = -\mu$. Our crime-free equilibrium is stable if all of these eigenvalues are negative. Since our parameters are all positive, we can automatically conclude that $\lambda_1 < 0$, $\lambda_3 < 0$ and $\lambda_4 < 0$. For λ_2 , we see that the condition $(\mu + \phi)(\mu + \omega) > \beta \mu N$ must be met in order for it to be negative. In other words, $\lambda_2 < 0$ if $R_0 < 1$. Thus, if $R_0 < 1$, the crime-free equilibrium is stable; otherwise, it is unstable.

Lemma 3. *For the One-Strike model, if $R_0 < 1$, then the crime-free equilibrium is locally asymptotically stable. Since the One-Strike model is a special case of the Three-Strikes model, a backwards bifurcation cannot exist for the One-Strike model.*

4 Parameter Estimation

In sociological models, it is difficult to find parameter values from published data. This inherent difficulty was especially evident when estimating values for β_i , where $i = 0, 1$

and 2, in the Three-Strikes model. Due to the length of the derivation for these values of β_i , we only included a short explanation for each β_i . Explicit calculations for the β_i 's can be found in the Appendix. From various technical reports we were able to calculate values for ϕ and ψ_j , where $j=1$ and 2. For parameters that could not be determined from published data, such as ϵ_i and ω_i , we state the assumptions made and the corresponding estimated values. These parameter estimates are then tabulated according to the model they represent. Presented first are parameter values for the Three-Strikes model since the parameters used in both the Infinite-Strikes Model and the One-Strike model are comparable to those in the Three-Strikes model.

4.1 Three-Strikes Model Parameter Estimates

$\mu \approx 0.03226 \text{ year}^{-1}$. Given that our population consists of individuals between the ages of 18 and 49, we assume that on average a person spends $\frac{1}{49-18} = \frac{1}{31} \approx 0.03226 \text{ years}^{-1}$ in our system. Due to the fact that individuals move quickly through our system, the natural death rate is negligible.

$\phi \approx 0.776 \text{ year}^{-1}$. ϕ , the arrest rate, was given as a projected value in [7].

$\delta \approx 0$. We assume that the rate at which individuals are sent to I_3 , i.e. given life sentences, on the first or second strike is negligible.

$\psi_1 \approx \frac{1}{3.5} \text{ year}^{-1}$. The average amount of time an individual is incarcerated for the first strike is 3.5 years [5]. Therefore, an individual leaves I_1 at a rate of $\frac{1}{3.5} \text{ year}^{-1}$.

$\psi_2 \approx \frac{1}{8} \text{ year}^{-1}$. The average amount of time an individual is incarcerated for the second strike is 8 years [5]. Therefore, an individual leaves I_2 at a rate of $\frac{1}{8} \text{ year}^{-1}$.

$\omega_0 \approx 0.03226 \text{ year}^{-1}$. We assume that a person not yet involved in criminal activity has no particular motivation to permanently remove himself from criminals, i.e. one is just as likely to leave the system by "reform" as by death. As such, we let ω_0 be equal to μ .

$\omega_1 \approx 0.6 \text{ year}^{-1}$. We base our calculations for ω_1 (and ω_2) on data from [7] and [2], and our estimated breakdown of the percentages of the population choosing to reform rather than to re-enter criminal activity.

$\omega_2 \approx 0.619 \text{ year}^{-1}$. See above.

$$\beta_0 \approx 2.9034 \times 10^{-7} \frac{1}{\text{people} \times \text{year}};$$

$$\beta_1 \approx 7.24 \times 10^{-6} \frac{1}{\text{people} \times \text{year}};$$

$$\beta_2 \approx 4.02 \times 10^{-6} \frac{1}{\text{people} \times \text{year}};$$

Each value of β_i , where $i= 0,1$, and 2, was calculated using data from [7] and [2]. We estimated these parameters using Los Angeles County offence rates and crime rates. These were rather complicated calculations, so the full derivation can be found in the Appendix.

$\epsilon_i \approx 0.0001 \text{ year}^{-1}$, where $i = 0,1$ and 2. Due to the fact that our initial population is large, the deterrent effect caused by the number of individuals in I_3 must be small. These values of ϵ_i are arbitrary.

A concise table of parameter values for the Three-Strikes Model is found in Table 2.

Parameter	Three-Strike Value	Reference
β_0	$2.9034 \times 10^{-7} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
β_1	$7.24 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
β_2	$4.02 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
ω_0	$0.03226 \text{ year}^{-1}$	see text
ω_1	0.6 year^{-1}	[4] [3]
ω_2	0.619 year^{-1}	[4] [3]
ψ_1	$\frac{1}{3.5} \text{ year}^{-1}$	[5]
ψ_2	$\frac{1}{8} \text{ year}^{-1}$	[5]
ϵ_0	0.0001 year^{-1}	see text
ϵ_1	0.0001 year^{-1}	see text
ϵ_2	0.0001 year^{-1}	see text
ϕ	0.776 year^{-1}	[7]
δ	0 year^{-1}	see text
μ	$0.03226 \text{ year}^{-1}$	see text

Table 2: Estimated Parameter Values For Three-Strikes Model

4.2 Infinite-Strikes Model Parameter Estimates

$\mu \approx 0.03226 \text{ year}^{-1}$. See above.

$\phi \approx 0.776 \text{ year}^{-1}$. See above.

$\psi \approx \frac{1}{3.5} \text{ year}^{-1}$. ψ in the Infinite-Strikes model is equal to ψ_1 in the Three-Strikes Model. In California’s Three-Strikes policy, the first “striking” offence is not punished by sentence enhancements. Therefore, the average time an individual serves for each offence in the Infinite-Strikes model is approximately equal to the amount of time a “first striker” serves.

$\omega \approx 0.7 \text{ year}^{-1}$. We base our calculation for ω on data from [7] and [2], and our estimated percentage of the population that chooses to reform rather than to re-enter criminal activity.

$\beta_0 \approx 2.9034 \times 10^{-7} \frac{1}{\text{people} \times \text{year}}$. β_0 in the Infinite-Strikes model is equal to β_0 in the Three-Strikes model. Due to the nature of our derivation of β_0 , β_0 has no explicit meaning by itself. It is within the context of $\beta_0 C$ that β_0 makes sense. See Appendix.

$\beta_1 \approx 6.203 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$. We base our calculations for β_1 on data from [7] and [2].

A concise table of these values is found in Table 3.

Parameter	Infinite-Strike Value	Reference
β_0	$2.903 \times 10^{-7} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
β_1	$6.203 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
ω	0.7 year ⁻¹	see text
ψ	1/3.5 year ⁻¹	[5]
ϕ	0.776 year ⁻¹	[7]
μ	0.03226 year ⁻¹	see text

Table 3: Estimated Parameter Values For Infinite-Strikes Model

4.3 One-Strike Model Parameter Estimates

$\mu \approx 0.03226 \text{ year}^{-1}$. See above.

$\phi \approx 0.776 \text{ year}^{-1}$. See above.

$\omega \approx 0.254 \text{ year}^{-1}$. We base our calculations for β_1 on data from [7] and [2]. ω in the One-Strike model is theoretical; we determine this particular value based on the assumption that approximately 75 percent of the susceptible population will leave by reforming as opposed to leaving through death or joining the criminal class.

$\beta \approx 4.07 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$. We base our calculations for β on data from [7] and [2].

$\epsilon \approx 0.0001 \text{ year}^{-1}$. See Three-Strike parameter estimates.

A concise table of these values is found in Table 4.

Parameter	One-Strike Value	Reference
β	$4.07 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}$	[7] [2]
ω	0.254 year ⁻¹	see text
ϵ	0.0001 year ⁻¹	see text
ϕ	0.776 year ⁻¹	[7]
μ	0.03226 year ⁻¹	see text

Table 4: Estimated Parameter Values For One-Strike Model

5 Numerical Simulations

5.1 Population Size and the Three-Strikes Model

It is important to examine the effect of the total population size N on the dynamics of our model(s). For the purpose of this analysis we will use the Los Angeles County parameters as our “base” parameters, i.e., the set of parameters will not change during our examination of different population sizes. We know that in the Three-Strikes model, $R_0 = (\frac{\beta_0 N}{\phi + \delta + \mu})(\frac{\mu}{\omega_0 + \mu})$ and that the crime-free equilibrium becomes unstable for $R_0 > 1$. Thus when $N > \frac{(\omega_0 + \mu)(\phi + \delta + \mu)}{\beta_0 \mu} \equiv N^*$ the crime-free equilibrium is unstable, while if $N < N^*$ the C.F.E. is stable. By substituting the Los Angeles County values for μ , β_0 , ϕ , δ and ω_0 in the Three-Strikes model, we can determine $N^* \approx 5567679$. For $N > N^*$, the model will exhibit behavior with an increasing total population leading to an increase in the number of people who end up in prison as t approaches infinity. Figure 5 shows the total incarcerated and reformed populations for $N=6$ million, 8 million, and 10 million. We introduce one criminal into a fully susceptible population. The thickest curves represent the prison population ($I_1 + I_2 + I_3$), which increases with the total population.

We estimate a range for values of N that satisfy the conditions for the existence of two endemic equilibria—when $R_0 < 1$, i.e. ($N < N^*$). By plotting our solutions and introducing one criminal into a fully susceptible population each time and decreasing N downwards from N^* , we were able to determine a population value where only the crime-free equilibrium exists. We found this value, which we denote as N^c , to be approximately 2820000. For $N^c < N < N^*$, we were able to visually verify the existence of two endemic equilibria. Since $2820000 \approx 0.508 * 5567679$, we can conclude that the bifurcation from the Infinite-Strikes model also exists in the Three-Strikes model, with the saddle node bifurcation point approximately equal to 0.508. This visual analysis is shown in Figure 6—for N less than 2820000 the stable crime-free equilibrium is approached, while two endemic equilibria exist for greater values of N .

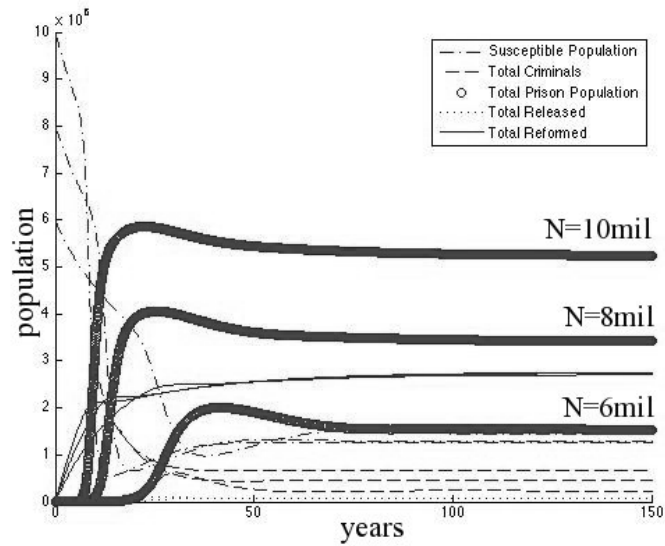


Figure 5: Solutions vs. Time for $N = 6$ million, 8 million and 10 million

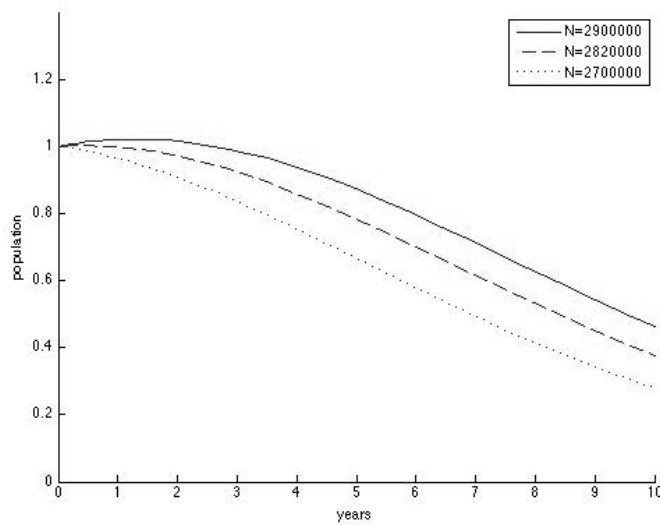


Figure 6: Total criminal population for N near N^c

The Three-Strikes model exhibits a general pattern of behavior for $N < N^c$, which is shown below for an example population of 2 million (with one criminal introduced initially):

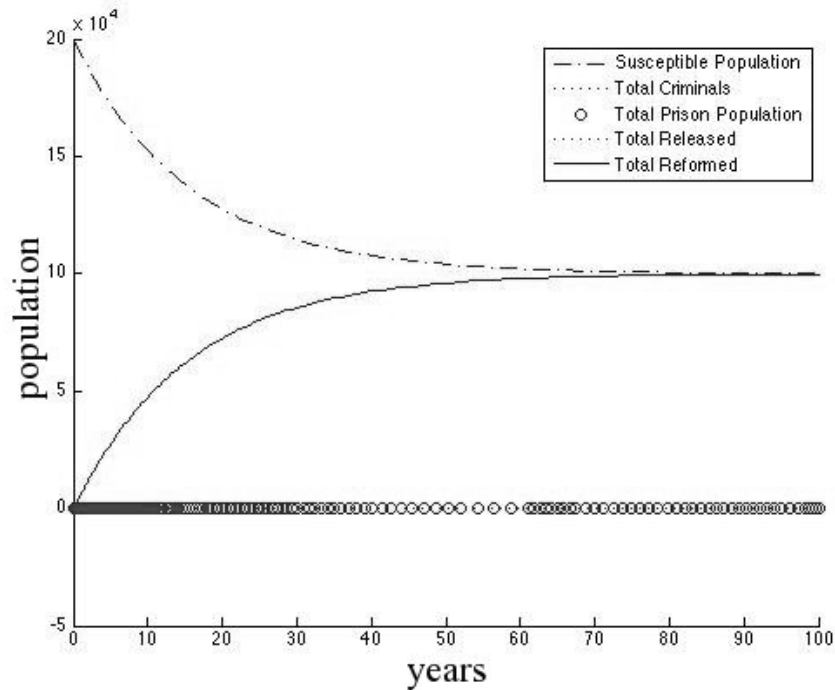


Figure 7: Solutions versus time for $N=2000000$ (1 criminal introduced)

The system approaches the stable crime-free equilibrium (with prison populations dying out as a result), with the susceptible and reformed populations approaching the same value as t goes to infinity.

5.2 Analysis of the Three-Strikes Model and Infinite-Strikes Model in Los Angeles County

In this section, we will numerically examine the effects of both the Three-Strikes model and Infinite-Strikes model on Los Angeles County, which has a population of ≈ 10 million.

In the previous section we showed the results of introducing one criminal into a fully susceptible population of 10 million people under the Three Strikes Law. Figure 8 below shows the total incarcerated, reformed and criminal populations from this particular simulation under the Three-Strikes policy, and the total incarcerated, reformed and criminal populations as a result of one criminal introduced into a fully susceptible population under the Infinite-Strikes policy.

From these graphs, we see that the Three-Strikes model is more effective at controlling the level of crime in a fully susceptible population than the Infinite-Strikes model (ap-

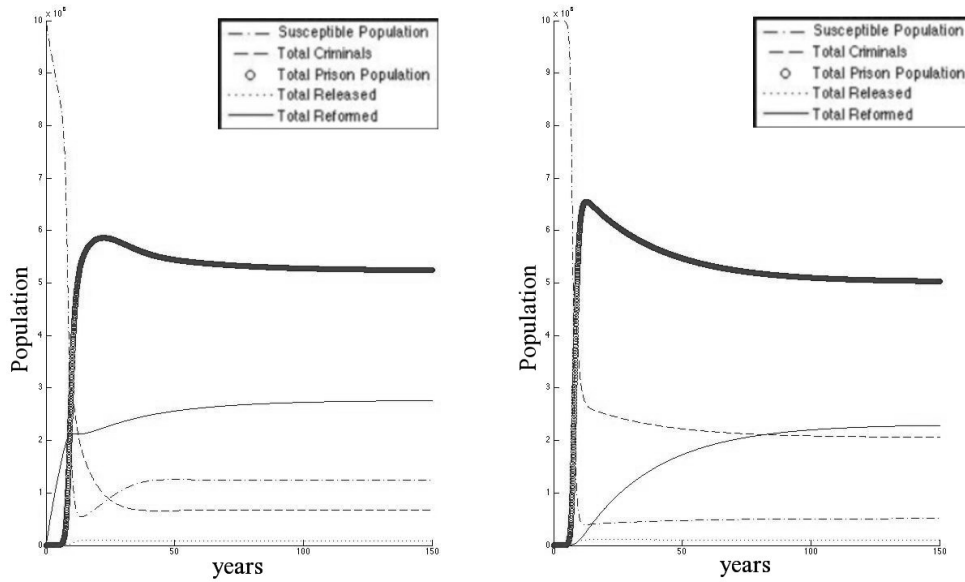


Figure 8: Total population in the S , C , I , R and V classes: Three-Strikes (left), Infinite-Strikes (right)

proximately 30 percent of the criminals present under the Infinite-Strikes model are present under the Three-Strikes model), and also leads to a higher reformed population (approximately 22 percent higher). The Three-Strikes model has the effect of sending slightly more of the population (approximately 3 percent more of the total population) to prison in the long term. The I_3 class, in particular, is shown in Figure 9:

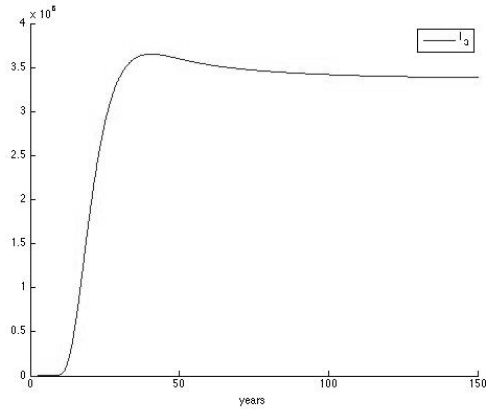


Figure 9: I_3 population under the Three-Strikes Law

We give a projection for the future of the system in LA under the Three-Strikes Law. Using the same parameters and the current class data for LA County [1], we can run our model from the present time (we use a high initial "reformed" population to reflect the high percentage of the current LA population currently removed from criminal activity) to obtain the following solutions:

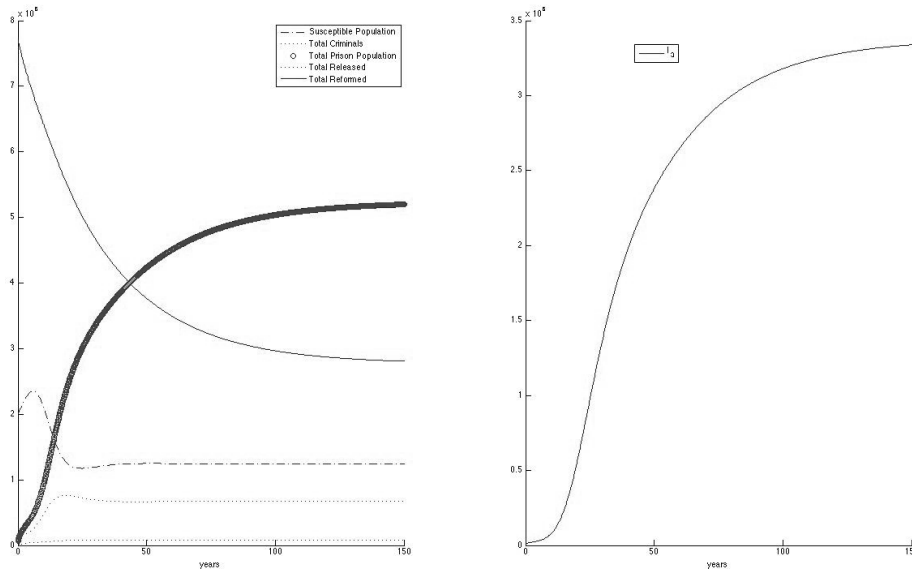


Figure 10: Our model’s prediction for LA under the Three-Strikes Law (with high initial reformed population): Totals (left) and I_3 (right)

Our figures indicate that while the current situation in Los Angeles is reasonable, as time goes on, the deterrent effect of the Three-Strikes system will not be enough to prevent an unreasonably high proportion of the population from being sent to prison. In addition, the reformed class will not be able to sustain itself over time.

In comparison, we can run a simulation in which the Infinite-strikes policy is reinstated in LA county with the current class populations:

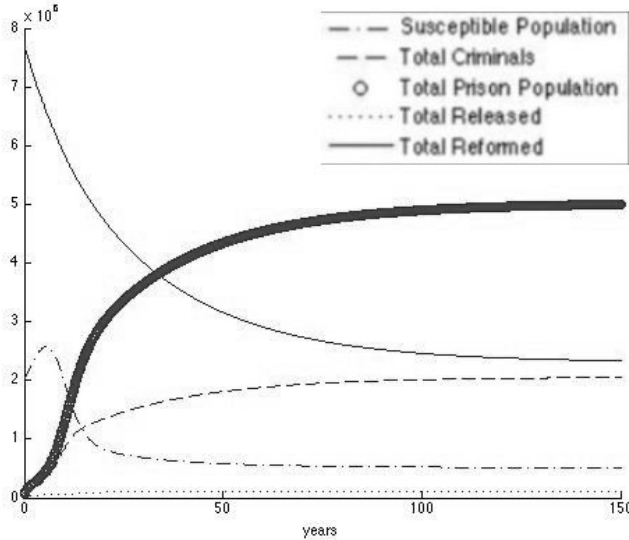


Figure 11: Our model’s prediction of LA under a reinstated Infinite-Strikes policy (with high initial reformed population)

The figure indicates that the Infinite-strikes policy will be less effective at controlling the amount of crime in the population than the Three-Strikes Law, and lacking the deterrent effect of the latter, does a worse job of maintaining the number of individuals in the reformed class. We see that as time goes on, the total prison population is less than that under the Three-Strikes Law (approximately 4997300 at equilibrium, versus 5197900 at equilibrium in the Three-Strikes model). However, considering the high criminal population in the long run under the Infinite-Strikes policy, this 200000-person difference (2 percent of the population) can be seen as justifiable.

5.3 The Three-Strikes Model in Other Regions

In this section, we will briefly examine the consequences of the Three-Strikes Law in two other states which implement the policy— New Mexico and Montana. Los Angeles County alone has a population of around 10 million; this is in sharp contrast to the populations of New Mexico and Montana— around 1.9 million and 900,000, respectively. Our earlier study on population size and the Three-Strike Law gives some indication as to how our model should behave in these sparsely-populated states. To examine the effects of the law, we will keep all of the parameters the same as those used for LA County, but will vary ϕ (the arrest rate) according to the data found in [1]. We estimate $\phi \approx .996 \text{ year}^{-1}$ in New Mexico and $\phi \approx .59 \text{ year}^{-1}$ in Montana [1]. Setting the initial reformed population equal to zero, we plot our hypothetical solutions using estimated current initial class data from both

states to obtain:

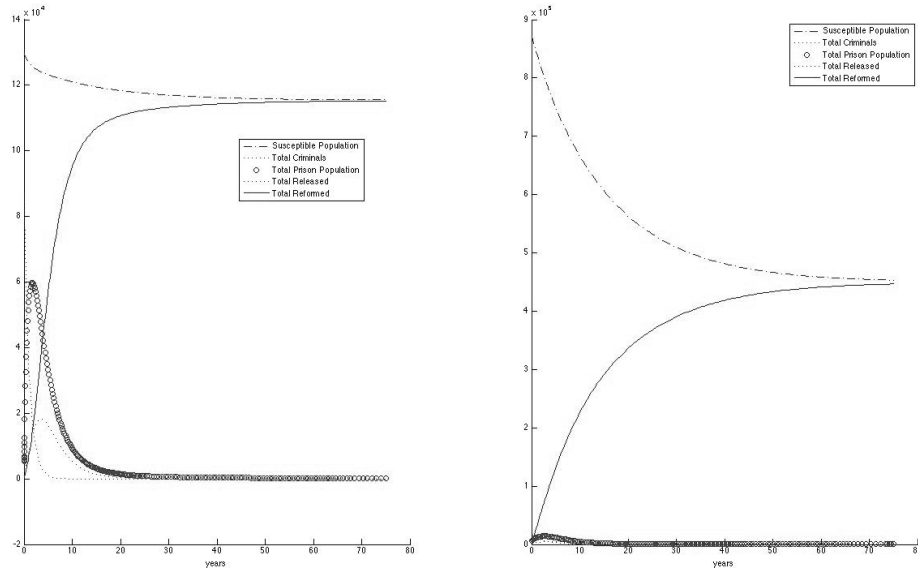


Figure 12: Our model's prediction for New Mexico (left) and Montana (right) under the Three-Strikes Law

These two examples show that the Three-Strikes Law, while not ideal in a heavily populated area such as LA County, is very effective at both controlling crime and keeping prison populations low in sparsely-populated areas. We must see how these figures compare to solutions for the Infinite-Strikes model for New Mexico and Montana. These results are shown below:

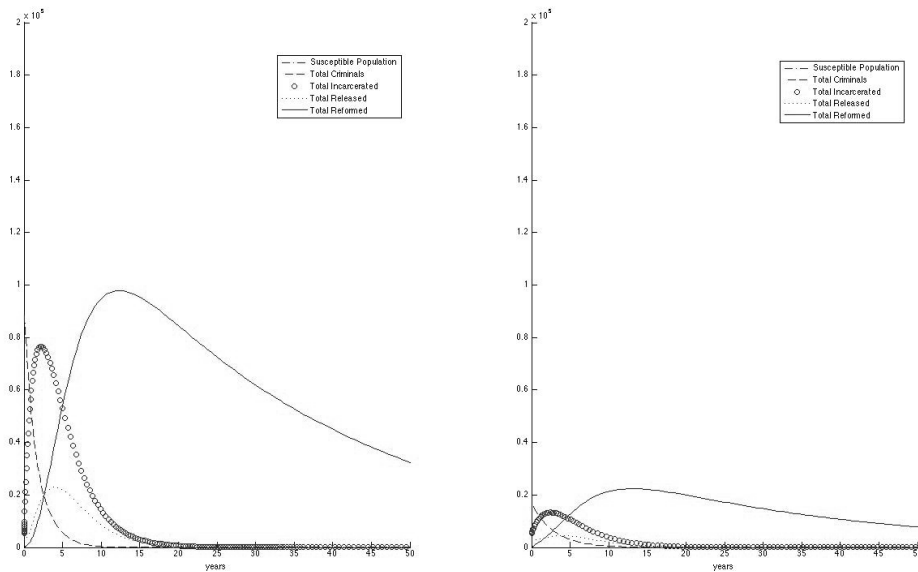


Figure 13: Our model’s prediction for New Mexico (left) and Montana (right) under the Infinite-Strikes model

From these graphs, we see that criminal activity actually dies out more quickly under the Infinite-Strikes model than under the Three-Strikes model (criminal activity disappears at around 10 years under the former, vs. around 20 years under the latter). This is due to the shorter sentencing and minimal amount of reappearances in the C classes (due to the decreased number of contacts) found in the Infinite-Strikes model. We conclude from these figures that the Infinite-Strikes model is better at controlling crime than the Three-Strikes model in sparsely populated areas; however, it is interesting to note that while the behavior of the two models is very different for large population sizes, they both tend toward the same behavior as population size decreases. The Three-Strikes model creates a much higher reformed population in Montana and New Mexico, but once crime has been eradicated, the number of people in the reformed class can be seen as irrelevant (since no one from the reformed or susceptible classes is being drawn into criminal activity.)

5.4 Comparison to the Hypothetical One-Strike Model

Finally, in order to address the question of whether an optimum number of strikes exists for a population as large as that of LA county, we will compare the results given by the Three-Strikes and Infinite-Strikes models to the results found by introducing one criminal into a fully susceptible population of 10 million under our hypothetical One-Strike model.

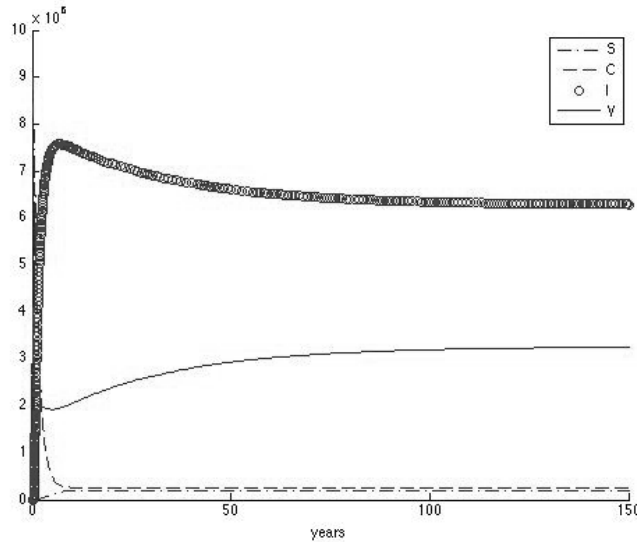


Figure 14: A population of 10 million under the One-Strike model

As one would expect, the One-Strike model controls crime very well, but at the cost of sending the majority of the population into prison. In the long term, it is much less effective compared to the Three-Strikes policy due to the fact that there are no opportunities for those involved in criminal activity to reform.

We can now delve into the question of what is the optimum number of strikes. Ideally, we want to maximize the population in the reformed class and minimize the populations in the criminal and incarcerated classes. In terms of controlling crime in a large urban population, the One-Strike model is the most effective, followed by the Three-Strikes model and the Infinite-Strikes model. In terms of having the highest number of reformed individuals, the Three-Strikes model is first, followed by the Infinite-Strikes model and the One-Strike model. In terms of having the lowest number of incarcerated individuals, the Infinite-Strikes model is first, followed by the Three-Strikes model and the One-Strike model. Based on these results for the Three-Strikes model, the Infinite-Strikes model and the One-Strike model, we hypothesize that the optimum number of strikes in a large urban population is likely greater than three, but finite.

6 Results and Conclusions

Having performed the various numerical simulations, we now give an overall summary of our results. We show that a backwards bifurcation exists for the Infinite-Strikes model, thereby implying that criminality can arise even when the basic reproductive number is less than one. We then conjecture that a backwards bifurcation is also possible for the

Three-Strikes model based on the assumption that the Infinite-Strikes model encompasses the Three-Strikes model in its framework. However, we are unable to rigorously verify this claim due to the complex nature of the equations that arise from the analysis. Thus, we have to settle for a simple graphical check that there is indeed a saddle-node bifurcation point. The existence of this saddle-node bifurcation point implies that a backwards bifurcation exists for the Three-Strikes model.

From our numerical simulations, we can conclude that the Three-Strikes model is more effective at controlling crime than the Infinite-Strikes model in densely populated areas like Los Angeles County. In other words, the endemic equilibrium value of the number of criminals in the Three-Strikes model is significantly lower than the endemic equilibrium value of the number of criminals in the Infinite-Strikes model. In addition, the Three-Strikes model exhibits a higher endemic equilibrium value for the reformed population than the value in the Infinite-Strikes model.

However, we see in Figures 10 and 11 that these population levels in the reformed classes for both models cannot be sustained over time. All the while, the prison population rises and actually intersects the total reformed population at a particular year. For the Three-Strikes model, the intersection occurs at approximately 32.3 years, whereas for the Infinite-Strikes model, the intersection occurs at approximately 28.0 years. From this, we can conclude that even though the reformed population is declining in both models, the Three-Strikes model still manages to keep a higher level of reformed individuals than imprisoned individuals for a longer period of time than the Infinite-Strikes model. In fact, the Three-Strikes model achieves a greater number of reformed individuals at this intersection than the Infinite-Strikes model. Hence, we see that the Three-Strikes model is more effective at deterring crime than the Infinite-Strikes model.

However, the Three-Strikes model has a tendency to simply incapacitate criminals, as shown in Figure 8. This effect is made more clear in Figure 10, as the population in I_3 approaches an endemic equilibrium value that is significantly higher than the endemic equilibrium value for the reformed population. We see that the Infinite-Strikes model also manifests this “incapacitation” effect but to a lesser degree. Thus, the Infinite-Strikes model is better than the Three-Strikes model in the sense that the Infinite-Strikes model sends less of the population to prison over time. However, this benefit is only marginal when compared to the effective deterrent effect of the Three-Strikes system on densely populated regions. Overall, in a densely populated area like Los Angeles County, the Three-Strikes system is better able to deter crime than the Infinite-Strikes model, but at the cost of imprisoning more people. Ideally, we want a policy that deters crime but minimizes the number of incarcerated individuals. Hence, the Three-Strikes system does not work perfectly in a dense, urban population, but is still more effective than the Infinite-Strikes model.

We then examine the effect of the Three-Strikes Law on sparsely population regions, specifically in New Mexico and Montana. We find that the endemic equilibrium level for the reformed population is drastically higher than the equilibrium level for the imprisoned population and the criminal population for both states. In fact, the imprisoned population goes to zero for both states (see Figure 12), which is consistent with actual statistics for

each state; New Mexico and Montana have never sentenced a “third striker” [13]. Hence, the Three-Strikes system is more effective in areas that are thinly populated than in areas that are densely populated. The reason for this lies in the contact processes involved in the Three-Strikes model. Since people in New Mexico and Montana are more spread out than people in Los Angeles County, fewer contacts occur between people in general, but especially between criminals and the susceptible population. Hence, the number of criminals eventually reaches zero, as does the imprisoned population.

The same effect can be seen when we look at New Mexico and Montana under the Infinite-Strikes system (Figure 13). Both the incarcerated and criminal populations die off eventually, so like the Three-Strikes system, the Infinite-Strikes system is more effective at deterring crime in a sparsely populated area than in a densely populated area. When we compare both models however, we see that the criminal population approaches zero faster under the Infinite-Strikes system than the Three-Strikes system. In fact, criminal activity disappears at around 10 years for the Infinite-Strikes system, but at around 20 years for the Three-Strikes system. Therefore, the Infinite-Strikes system is a more efficient means to deter crime than the Three-Strikes system in a sparsely populated area.

Our final analysis consists of a comparison between the Three-Strikes model and the Infinite-Strikes model to the One-Strike model. We find that the One-Strike model controls crime very effectively, but at the cost of sending almost everyone in the population to prison as shown in Figure 14. The reformed population is also very small relative to the size of the imprisoned population. Hence, the One-Strike model is a more stringent form of the Three-Strikes model, which is expected. Due to this severity of the One-Strike system, no opportunities are given to criminals to reform, thus making a One-Strike policy an extremist approach to law enforcement. Similarly, a Two-Strikes system will be more strict than the Three-Strikes system, but less effective overall. Therefore, the optimal policy will most likely consist of more than three strikes. Further research must be conducted in order to determine the maximum number of strikes needed to effectively deter crime in densely populated urban areas. Other areas for further development may include the effect of the ϵ 's on the Three-Strikes system. Variation in the ϵ 's may cause differing equilibrium populations in the incarcerated, reformed and criminal classes. Considering these models from a standard incidence point of view may also produce different results. In doing so however, one will not be able to analyze the impact of population density on crime rates.

7 Acknowledgements

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8 Appendix

8.1 \mathcal{F} and \mathcal{V} for the Calculation of R_0 for the Three-Strikes Model

$$\mathcal{F} = \begin{bmatrix} 0 \\ \beta_0 CS \\ \beta_1 CR_1 \\ \beta_2 CR_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } \mathcal{V} = \begin{bmatrix} -\mu N + \beta_0 CS + S \left(\omega_0 + \frac{\epsilon_0 I_3}{N} \right) + \mu S \\ \phi C_0 + \delta C_0 + \mu C_0 \\ \phi C_1 + \delta C_1 + \mu C_1 \\ \phi C_2 + \mu C_2 \\ -\phi C_0 + \psi_1 I_1 + \mu I_1 \\ -\phi C_1 + \psi_2 I_2 + \mu I_2 \\ -\phi C_2 - \delta C_0 - \delta C_1 + \mu I_3 \\ -\psi_1 I_1 + \beta_1 CR_1 + R_1 \left(\omega_1 + \frac{\epsilon_1 I_3}{N} \right) + \mu R_1 \\ -\psi_2 I_2 + \beta_2 CR_2 + R_2 \left(\omega_2 + \frac{\epsilon_2 I_3}{N} \right) + \mu R_2 \\ -S \left(\omega_0 + \frac{\epsilon_0 I_3}{N} \right) - R_1 \left(\omega_1 + \frac{\epsilon_1 I_3}{N} \right) - R_2 \left(\omega_2 + \frac{\epsilon_2 I_3}{N} \right) + \mu V \end{bmatrix}$$

8.2 Alternative Set of Endemic Parameter Conditions for the Infinite-Strikes Model

Here we present another set of parameter conditions that is equally valid in assessing the dynamics of the endemic equilibria in the Infinite-Strikes model.

Theorem 2.

(i) If $R_0 > 1$, then there is only one endemic equilibrium.

(ii) Suppose $R_0^+ < R_0 < 1$ and $R_1 > \frac{1}{1-K}$, where $K = 1 - \left(\frac{\phi}{\mu+\phi} \right) \left(\frac{\psi}{\mu+\psi} \right)$ and $R_0^+ = \frac{R_1+1+2\sqrt{R_1(1-K)}}{\frac{(R_1-1)^2}{R_1K}+4}$. Then there exist two endemic equilibria. Otherwise, there is none.

Proof. The first implication is trivial. The fact that $R_0 > 1 \Leftrightarrow 0 < c = f(0)$ and $a < 0$, where f is the quadratic equation in (1), automatically guarantees the existence of precisely one endemic equilibrium, since $f(1) < 0$.

For the second implication, recall that if $R_0 < 1 \Leftrightarrow c < 0$, $b > 0$ and $b^2 - 4ac > 0$, then there are two endemic equilibria. Let us first consider the condition $b > 0$. Instead of factoring out b_1 as before, let us factor out b_0 . Hence, $b > 0 \Rightarrow$

$$\begin{aligned} b_0 \left(b_1 - \frac{(\mu + \phi)(\mu + \omega_1)}{\mu} \right) - b_1 \mu \left(1 + \frac{\mu}{\mu + \psi} \right) &> 0 \\ \Rightarrow b_1 &> \frac{(\mu + \phi)(\mu + \omega_1)}{\mu} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{b_1\mu}{(\mu + \phi)(\mu + \omega_1)} > 1 \\ &\Rightarrow R_1 > 1. \end{aligned}$$

In addition, we also need

$$\begin{aligned} b_0 \left(b_1 - \frac{(\mu + \phi)(\mu + \omega_1)}{\mu} \right) &> b_1\mu \left(1 + \frac{\phi}{\mu + \psi} \right) = b_1K(\mu + \phi) \\ &\Rightarrow b_0 \left(\frac{b_1\mu - (\mu + \phi)(\mu + \omega_1)}{\mu} \right) > b_1K(\mu + \phi) \\ &\Rightarrow b_0 \left(\frac{(R_1 - 1)(\mu + \phi)(\mu + \omega_1)}{\mu} \right) > b_1K(\mu + \phi) \\ &\Rightarrow R_0 > \left(\frac{R_1}{R_1 - 1} \right) K. \end{aligned}$$

Hence, it seems we have an extra parameter condition for R_0 . However, once we reduce the condition $b^2 - 4ac > 0$ to its parameter conditions, we see that this condition $R_0 > \left(\frac{R_1}{R_1 - 1} \right) K$ goes away. From the last part of section 3.5, we already know that

$$\begin{aligned} R_0^+ &= \frac{R_1 + 1 + 2\sqrt{R_1(1 - K)}}{\frac{(R_1 - 1)^2}{R_1K} + 4} \\ R_0^- &= \frac{R_1 + 1 - 2\sqrt{R_1(1 - K)}}{\frac{(R_1 - 1)^2}{R_1K} + 4}. \end{aligned}$$

Once again, $b^2 - 4ac > 0$ is satisfied \Leftrightarrow

- (i) $0 < R_0 < R_0^-$ or
- (ii) $R_0 > R_0^+$

We want to show that $R_0^+ > \left(\frac{R_1}{R_1 - 1} \right) K > R_0^-$ in order to reduce the total number of conditions down to just one. Let us consider the second equality first. Notice we only need to show $K > R_0^-$ in order for the inequality to be satisfied since $R_1 > 1 \Rightarrow \left(\frac{R_1}{R_1 - 1} \right) K > K$. Proceeding like before in the case of $R_0^+ > K$, we have $R_0^- < K \Leftrightarrow$

$$\begin{aligned} R_1 + 1 - 2\sqrt{R_1(1 - K)} &< \frac{(R_1 - 1)^2}{R_1} + 4K \\ R_1^2 + R_1 - 2R_1\sqrt{R_1(1 - K)} &< (R_1 - 1)^2 + 4KR_1 \\ 2R_1\sqrt{R_1(1 - K)} &> 3R_1 - 1 - 4KR_1 \\ 4R_1(1 - K) &> \left(3 - \frac{1}{R_1} - 4K \right)^2 \\ 4(1 - K)R_1^3 - (3 - 4K)^2R_1^2 + 2(3 - 4K)R_1 - 1 &> 0. \end{aligned}$$

Let $p(R_1) \equiv 4(1-K)R_1^3 - (3-4K)^2R_1^2 + 2(3-4K)R_1 - 1$ and set $p(R_1) = 0$. The roots of p are then $R_1^a = \frac{1}{4(1-K)}$, $R_1^b = 1 - 2K + 2\sqrt{-K(1-K)}$ and $R_1^c = 1 - 2K - 2\sqrt{-K(1-K)}$. Since $K < 1$ by definition, R_1^b and R_1^c are imaginary, so we only consider R_1^a in our analysis. Specifically, we have the following two cases to analyze:

$$(a) \quad R_0^- < K \Leftrightarrow p(R_1) > 0 \Leftrightarrow 3 - 4K - \frac{1}{R_1} < 0 < 2\sqrt{R_1(1-K)}$$

$$(b) \quad R_0^- < K \Leftrightarrow p(R_1) > 0 \Leftrightarrow 0 < 3 - 4K - \frac{1}{R_1} < 2\sqrt{R_1(1-K)}$$

Consider the first case. We see that $3 - 4K - \frac{1}{R_1} < 0 \Leftrightarrow 3 - 4K < \frac{1}{R_1}$ which is true if and only if $K \geq \frac{3}{4}$. From the second case, we have that $p(R_1) > 0 \Leftrightarrow R_1 > R_1^a = \frac{1}{4(1-K)}$. This last inequality is automatically satisfied if $K < \frac{3}{4}$, since $K < \frac{3}{4} \Leftrightarrow \frac{1}{4(1-K)} < 1$ so then $R_1 > 1 > \frac{1}{4(1-K)}$. Therefore, the condition $R_0^- < K$ is satisfied for all possible values of K , where $0 < K < 1$.

We now need to show $R_0^+ > \left(\frac{R_1}{R_1-1}\right)K$. First, we have $R_0^+ > \frac{R_1K}{R_1-1} \Leftrightarrow$

$$\begin{aligned} 1 + R_1 + 2\sqrt{R_1(1-K)} &> 4K \left(\frac{R_1}{R_1-1}\right) + R_1 - 1 \\ \sqrt{R_1(1-K)} &> \left(\frac{2KR_1}{R_1-1} - 1\right) \end{aligned} \quad (10)$$

(10) is true if and only if either

$$(a) \quad \frac{2R_1K}{R_1-1} - 1 < 0 \text{ since } \sqrt{R_1(1-K)} > 0 \text{ for every } K.$$

$$(b) \quad 0 < \frac{2R_1K}{R_1-1} - 1 \text{ and } R_1(1-K) > \left(\frac{2R_1K}{R_1-1} - 1\right)^2$$

Reducing the condition in (a), we have an equivalent condition in the form of $R_1 > \frac{1}{1-2K}$ and $0 < K < \frac{1}{2}$. In (b), we have $0 < \frac{2R_1K}{R_1-1} - 1 \Leftrightarrow K \geq \frac{1}{2}$ and $R_1(1-K) > \left(\frac{2R_1K}{R_1-1} - 1\right)^2 \Leftrightarrow g(R_1) > 0$ where

$$g(R_1) = (1-K)R_1^3 - (3-6K+4K^2)R_1^2 + (3-5K)R_1 - 1$$

and is gotten from expanding (10). Setting $g(R_1) = 0$, we arrive at the following roots:

$$R_1^a = \frac{1}{1-K}$$

$$R_1^b = 1 - 2K + 2\sqrt{-K(1-K)}$$

$$R_1^c = 1 - 2K - 2\sqrt{-K(1-K)}.$$

We see that R_1^b and R_1^c are imaginary since $K < 1$. Hence, $g(R_1) > 0 \Leftrightarrow R_1 > \frac{1}{1-K}$. We then have $R_0^+ > \left(\frac{R_1K}{R_1-1}\right) \Leftrightarrow$

- (i) $0 < K < \frac{1}{2}$ and $R_1 > \frac{1}{1-2K}$ or
- (ii) $K \geq \frac{1}{2}$ and $R_1 > \frac{1}{1-K}$ or
- (iii) $0 < K < \frac{1}{2}$ and $R_1 < \frac{1}{1-2K}$ and $R_1 > \frac{1}{1-K}$

(i) and (iii) implies that $0 < K < \frac{1}{2}$ and $R_1 > \frac{1}{1-K}$ since $\frac{1}{1-2K} > \frac{1}{1-K}$ for $0 < K < \frac{1}{2}$. Combining this new condition with (ii), we have $R_1 > \frac{1}{1-K}$ for every K . Hence, $R_0^+ > \left(\frac{R_1 K}{R_1 - 1}\right) \Leftrightarrow R_1 > \frac{1}{1-K}$. The condition for the existence of two endemic equilibria is then $R_1 > 1$ and $\max(R_0^+, \frac{R_1 K}{R_1 - 1}) < R_0 < 1$. This condition can be separated into two cases:

- (I) $1 < R_1 < \frac{1}{1-K}$ and $\frac{R_1 K}{R_1 - 1} < R_0 < 1$
- (II) $R_1 > \frac{1}{1-K}$ and $R_0^+ < R_0 < 1$.

But (I) is impossible because if $R_1 < \frac{1}{1-K}$, then $\frac{R_1 K}{R_1 - 1} > \left(\frac{\frac{1}{1-K}}{K} K\right) = 1$. Thus, only (II) is possible. \square

8.3 Derivation of the β_i 's in the Three-Strikes Model

We estimate β_0, β_1 and β_2 in the Three-Strikes model using data from 2004-2005. The projected offense rate for 2000-2007 (given in $\frac{\text{offenses}}{\text{offender*year}}$) according to [7], is 0.4216 for violent crimes in Los Angeles County. We can divide the number of crimes (per year) in 2004 by this offense rate to obtain the estimated number of criminals (per year) in 2004:

$$\frac{76203 \frac{\text{offenses}}{\text{year}}}{0.4216 \frac{\text{offenses}}{\text{offender*year}}} = 180747 \text{ offenders}$$

This is equal to the number of offenders per year in 2004. We must now compute the number of criminals from 2004 remaining in 2005. This can be found by subtracting the number of criminals from 2004 being arrested from the total number. Denote this quantity by x (x has units $\frac{\text{offenders}}{\text{year}}$).

$$x = 180747 - \phi 180747 \approx 40487.$$

Next we compute the number of new appearances in C (per year). This can be found by subtracting x from the number of offenders/year in 2005:

$$\text{New Appearances in } C/\text{year} = \frac{66350}{0.4216} - 40487 \approx 116890.$$

We must now break down the number of new appearances in C/year into the number of new appearances in C_0/year , the number of new appearances in C_1/year and the number of new appearances in C_2/year . We estimate this breakdown based on the distribution of the current prison populations in I_1, I_2 and I_3 [6]. Using this data, we estimate that $C_1 \approx 0.326C_0$, $C_2 \approx 0.075C_0$ and determine that New Appearances in $C_0/\text{year} \approx 83433$,

New Appearances in C_1 /year ≈ 27199 and New Appearances in C_2 /year ≈ 6258 . Estimating the susceptible population in 2004 to be approximately 1589863 (based on the definition of the susceptible class as those in LA county living in poverty) and using 20781 and 8609 as the populations for R_1 and R_2 respectively, we can obtain β_0, β_1 and β_2 as follows:

$$\beta_0 = \frac{83433}{1589863*180747} \approx 0.00000029034 \frac{1}{\text{people} \times \text{year}}$$

$$\beta_1 = \frac{27199}{20871*180747} \approx 0.00000724 \frac{1}{\text{people} \times \text{year}}$$

$$\beta_2 = \frac{6258}{8609*180747} \approx 0.00000402 \frac{1}{\text{people} \times \text{year}}.$$

8.4 Derivation of ω_1 and ω_2 in the Three-Strikes Model

Using data from [2], we estimated that approximately 32.6 percent of the population released with one strike was choosing to remove themselves from criminal activity. Using this we derived the relation $\frac{\mu+\omega_1}{\mu+\omega_1+\beta_1 C} = 0.326$, which, by estimating $\beta_1 C \approx 1.309 \text{ years}^{-1}$ ([2] and [7]), we solve to find $\omega_1 \approx 0.6 \text{ years}^{-1}$.

Similarly, using the relation $\frac{\mu+\omega_2}{\mu+\omega_2+\beta_2 C} = 0.23$ and estimating $\beta_2 C \approx 0.7266 \text{ years}^{-1}$ ([2] and [7]), we solve to find $\omega_2 \approx 0.619 \text{ years}^{-1}$.

8.5 Derivation of β_1 in the Infinite-Strikes Model

β_1 in the Infinite-Strikes Model is equal to (New Appearances in C/year - New Appearances in C from S/year)/(R*C). Since the New Appearances in C/year from S is equal to $\beta_0 CS$ and we know the estimated number of people in R and C over the period of 2004-2005 from [2] and our earlier calculations, we can find β_1 as follows:

$$\beta_1 \approx \frac{116890 - (2.9034 \times 10^{-7})(180747)(1589863)}{(29480)(180747)} \approx 6.203 \times 10^{-6} \frac{1}{\text{people} \times \text{year}}.$$

8.6 Derivation of ω in the Infinite-Strikes Model

Similarly to our calculations for ω_1 and ω_2 in the Three-Strikes Model, using the relation $\frac{\mu+\omega}{\mu+\omega+\beta_1 C} = 0.556$ (using data from [7] and [2], we estimated that under the Infinite-Strikes policy, 55.6 percent of people do not return to criminal activity after being released from prison), we found $\omega \approx 0.7 \text{ years}^{-1}$.

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