Vaccination Strategy and Optimal Control for Seasonal and H1N1 Influenza Outbreak

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Abstract

During the spring of 2009, a novel H1N1 influenza virus posed a serious threat worldwide. However, seasonal influenza is still a public health concern since each year in the United States it kills 36,000 people, causes more than 200,000 hospitalizations, and infects up to 20% of the population. In this paper, we explore what could happen if both a seasonal influenza and H1N1 outbreak were to coincide. First, a simple SAIR model is considered to study the dynamics of a single influenza strain. Then, a more complex two-strain model is constructed to examine the dynamics of two coexisting strains. In the two-strain model, we incorporate a seasonal flu vaccine to study the impact of vaccination on these dynamics. Optimal control theory is applied to both the single-strain model and the two-strain model with the goal of reducing the overall morbidity during an outbreak. Two controls are introduced to the systems: social distancing and treatment of H1N1 infected individuals. Introducing the seasonal vaccine significantly reduces the number of seasonal influenza cases, yet moderately increases the number of H1N1 infections. In the two-strain model, the application of optimal controls has a substantial impact on the H1N1 and seasonal influenza dynamics.

1 Introduction

Influenza (flu) is one of the most prevalent diseases that cause high mortality in the world. There is a flu season every year in the Northern Hemisphere during the winter (from November to April) [**?**]. In the United States 5% to 20% of the population gets the seasonal flu and about 36,000 people die of flu-related causes. For seasonal influenza elderly people, young children, and people with certain health conditions are at high risk for serious flu complications.

Influenza strains can be divided into three main types A, B and C. Type A is considered the

most important in terms of epidemiology because it can infect both birds and mammals. Influenza A viruses are further subdivided based on two surface proteins - hemagglutinin (Hh) and neuraminidase (Nn). These subtypes are further divided into strains or comparatively minor variants. Type B is specific to only humans and tends to be more deadly. These two types (A and B) are the cause of seasonal flu epidemics every year in the United States. Type C isn't as prevalent as its counterparts and only causes a mild respiratory illness, which is not thought to cause epidemics. [**?**].

The influenza viruses are continuously changing over time due to antigenic shift and drift. Since the virus is changing so frequently it allows the virus to evade the immune system and therefore attack the body. When an individual is infected and recovers from a specific variant of influenza, he/she becomes immune to future infections of that strain but the previous infection does not provide immunity to other strains even if they belong to the same subtype. This makes vaccination production and vaccination programs a challenge for public health policy. A new vaccine must be designed and produced each year to protect against the seasonal flu that incorporates subtypes based on estimations of scientists and international surveillance. The vaccine is a trivalent inactive vaccine (TIV) that includes one influenza type B, one of A subtype H1N1 and one of A subtype H3N2 [**?**].

Historically, influenza A has caused several pandemics such as the deadly 1918 pandemic and the recent (2009) H1N1 (Swine Flu) pandemic. Pandemics can be the result of the appearance of new subtypes, like in the Swine Flu pandemic. In this case, laboratories found that this novel strain has two genes from flu viruses that normally circulate in pigs in Europe and Asia, but also has avian genes and human genes. This kind of reassortment of flu viruses of different species is a big concern for authorities of disease control because, as in the case of the recent H1N1 flu strain, it produces more severe epidemics or pandemics.

The 'Swine flu', or H1N1 influenza, surfaced in March 2009 in Mexico and spread worldwide [**?**]. This strain has caused mortality with 87% of the deaths in individuals between the ages of 5 and 59 [**?**]. Many fear that this variant will go through antigenic drift producing a second wave or outbreak similar to the 1918-1919 Spanish influenza pandemic that killed 40-50 million people worldwide [**?**]. Since influenza viruses are more infectious during cold months, it has been predicted that this H1N1 strain will begin to infect more people during the winter time, which is around the same time that seasonal flu will begin its course [**?**]. Many questions and concerns have been formulated, such as whether or not the seasonal flu and the new H1N1 virus circulating together will bring even more morbidity and mortality than either separately. Although the seasonal vaccine has a strain of H1N1, it has shown very little efficacy in reducing the cases of the novel H1N1 virus present in the Swine flu pandemic. The vaccine for the new H1N1 influenza strain is currently in production but it won't be available until October 2009 [**?**]. So, when the flu season starts, people should get the seasonal flu vaccine in order to protect themselves against infection with the seasonal strain. It has been suggested in the past that the very young (< 5) and elderly (65+) should be vaccinated since they are most susceptible to becoming infected with seasonal influenza [**?**].

There are limited amounts of vaccines available for distribution because the pharmaceutical companies cannot make the vaccines fast enough. In the United States, there are only enough resources to vaccinate about one-third of the population [**?**]. In order to be efficient with the allocation of vaccines, they should logically be given to the high-risk groups. Because so many cases of influenza fail to show flu-like symptoms (asymptomatics), it is likely that some individuals may opt to get the flu vaccine even though they have already been infected in the past and have immunity against reinfection. In these cases, the vaccine is considered wasted. Another type of cost associated with vaccination occurs when someone in the high-risk group for seasonal influenza chooses not to get the seasonal vaccine. This interferes with the control of the disease because the individual could contract the virus and as a result, will increase the economic costs and infectiousness in the population.

In this paper, we explored a vaccination strategy in which we ask the patient whether or not they have shown flu-like symptoms or if they have received the seasonal vaccine in the current year. If the individual replies "No", they will be administered the seasonal vaccine. We determined the total morbidity and the total number of wasted vaccines. We also applied optimal control theory to our model to minimize the total morbidity taking into account the cost of the strategies of social distancing and treatment of H1N1 strain. In Section 2, we introduce a SAIR model for seasonal flu and analyzed the effect of individuals recovering with and without showing symptoms. Optimal control was also applied to the SAIR model in order to reduce infections by producing numerical results and simulations. Section 3 introduces a two-strain model that incorporates the seasonal flu, H1N1 virus, and the seasonal flu vaccine. In Section 4, analysis and simulations are performed on the two-strain model. Sections 5 explains the optimal control that was applied to the two-strain model to generate numerical results.

2 SAIR Model

We first considered a simple SAIR (Susceptible(S)-Asymptomatic(A)-Symptomatic(I)-Recovered(R)) model for H1N1 influenza in which every infected individual first goes through an asymptomatic stage and either becomes symptomatic at a rate α or recovers without showing symptoms at a rate κ. Asymptomatic individuals play an important role in the spread of a virus because they are unaware of their infectiousness and hence, more likely to come into contact with others. Similarly, because we are less likely to treat asymptomatic individuals, the duration of infection persists until they recover naturally.

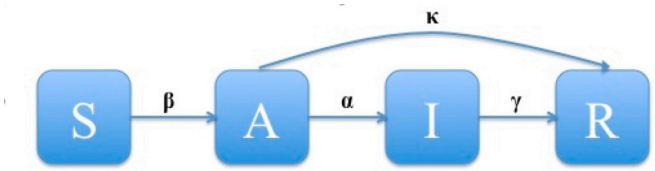


Figure 1: Diagram of the SAIR Compartmental Model

Table 1: List of Parameters

Symbol	Meaning
S	Susceptible
A	Asymptomatic
I	Symptomatic
R	Recovered
β	Transmission Rate
α	Rate an Individual Shows Symptoms
γ	Recovery Rate for Symptomatics
κ	Recovery Rate for Asymptomatics

2.1 Differential Equations

$$\frac{dS}{dt} = -\beta S\frac{A+I}{N} \tag{1}$$
$$\frac{dA}{dt} = \beta S\frac{A+I}{N} - (\alpha+\kappa)A \tag{2}$$
$$\frac{dI}{dt} = \alpha A - \gamma I \tag{3}$$
$$\frac{dR}{dt} = \gamma I + \kappa A \tag{4}$$
$$N = S + A + I + R \tag{5}$$

2.1.1 Calculating R_0 and the Final Size of SAIR

The basic reproductive number, R_0, can be defined as the number of secondary cases created in a fully susceptible population by one infective individual [**?**]. The approach we used to find the R_0 for this model was the next generation operator. The F matrix is defined by the newly infected individuals while the V matrix contains the newly symptomatic and recovered individuals. Tran-

sitions from the symptomatic and the vaccinated asymptomatic compartments are not considered to be new infections. Hence,

$$F = \begin{bmatrix} \frac{\beta S(A+I)}{N} \\ 0 \end{bmatrix} V = \begin{bmatrix} (\alpha+\kappa)A \\ -\alpha A + \gamma I \end{bmatrix}.$$

Taking the Jacobian of the matrices and evaluating them at the disease-free equilibrium $(N, 0, 0, 0)$, we obtain

$$F' = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix} V' = \begin{bmatrix} \alpha+\kappa & 0 \\ -\alpha & \gamma \end{bmatrix}.$$

Next, we take the inverse of V and multiply it by F'.

$$V'^{-1} = \begin{bmatrix} \frac{1}{\alpha+\kappa} & 0 \\ \frac{\alpha}{\alpha+\kappa}\frac{1}{\gamma} & \frac{1}{\gamma} \end{bmatrix},$$

$$F'V'^{-1} = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha+\kappa} & 0 \\ \frac{\alpha}{\alpha+\kappa}\frac{1}{\gamma} & \frac{1}{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\kappa} + \frac{\beta}{\alpha+\kappa}\frac{\alpha}{\gamma} & \frac{\beta}{\gamma} \\ 0 & 0 \end{bmatrix}.$$

If we examine the $F'V'^{-1}$ matrix, we notice that it is an upper triangular matrix. Thus, the eigenvalues are the diagonal entries and the basic reproduction number for the SAIR model is

$$R_0 = max[\lambda_1, \lambda_2],$$

where

$$\begin{aligned} \lambda_1 &= \frac{\beta}{\alpha+\kappa} + \frac{\beta}{\alpha+\kappa}\frac{\alpha}{\gamma}, \\ \lambda_2 &= 0. \end{aligned}$$

Hence

$$R_0 = \frac{\beta}{\alpha+\kappa} + \frac{\beta}{\alpha+\kappa}\frac{\alpha}{\gamma}.$$

The terms in R_0 represent the different ways of being infectious. Since β is the rate of entering A given an infectious contact, and $\frac{1}{\alpha+\kappa}$ is the length of time spent in A, $\frac{\beta}{\alpha+\kappa}$ describes an individual residing in the asymptomatic compartment. Similarly, because an individual must travel

through the asymptomatic compartment to become symptomatic, $\frac{\beta}{\alpha+\kappa}\frac{\alpha}{\gamma}$ represents being in the symptomatic class.

The final size relation of an epidemic model is an expression relating the basic reproductive number and the number of susceptible individuals at the end of the epidemic. We calculate the final size relation to gain information about the total morbidity at the end of a seasonal influenza outbreak in our SAIR model. We do this by finding an expression for $S_\infty = \lim_{t\to\infty} \mathrm{S(t)}$.

Let

$$\int_0^\infty (S^{'}(t) + A^{'}(t) + I^{'}(t))\, dt = -\kappa \int_0^\infty A(t)\, dt - \gamma \int_0^\infty I(t)dt, \tag{6}$$

and

$$\int_0^\infty (S^{'}(t) + A^{'}(t) + R^{'}(t))\, dt = -\alpha \int_0^\infty A(t)\, dt - \gamma \int_0^\infty I(t)dt. \tag{7}$$

Evaluating the left hand side of (**??**), we find that

$$\int_0^\infty (S^{'}(t) + A^{'}(t) + I^{'}(t))\, dt = S_\infty + A_\infty + I_\infty - S_0 + A_0 + I_0, \tag{8}$$

where $A_\infty = \lim_{t\to\infty} A(t)$, and $I_\infty = \lim_{t\to\infty} I(t)$ are 0 and $I_0 = 0$.
Therefore,

$$\int_0^\infty (S^{'}(t) + A^{'}(t) + I^{'}(t))\, dt = S_\infty - N. \tag{9}$$

Similarly from (6),

$$\int_0^\infty (S^{'}(t) + A^{'}(t) + R^{'}(t))\, dt = S_\infty - S_0 - A_0 + R_\infty. \tag{10}$$

Since $S_0 + A_0 = N$ and $R_\infty = N - S_\infty$, the right hand side of (**??**) reduces to $S_\infty - N + N - S_\infty = 0$ which implies,

$$\int_0^\infty I(t)\, dt = \frac{\alpha}{\gamma} \int_0^\infty A(t)dt. \tag{11}$$

If we substitute (**??**) into (**??**) we acquire

$$S_\infty - N = -(\alpha + \kappa) \int_0^\infty A(t)dt. \tag{12}$$

Dividing both sides of $S^{'} = -\beta S(t)\dfrac{A(t)+I(t)}{N}$ by S we develop

$$\frac{S'}{S} = -\beta\frac{A(t)+I(t)}{N}. \tag{13}$$

Integrating (**??**) and substituting (**??**) we get

$$\begin{aligned}\int_0^\infty \left(\frac{d}{dt}\ln(S)\right)dt &= -\frac{\beta}{N}\int_0^\infty (A(t)+I(t))dt, &(14)\\ &= -\frac{\beta}{N}\left(\frac{\alpha+\gamma}{\gamma}\right)\int_0^\infty A(t)dt. &(15)\end{aligned}$$

Since $\displaystyle\int_0^\infty \left(\frac{d}{dt}\ln(S)\right)dt = -\ln\left(\frac{S_0}{S_\infty}\right)$,

$$\int_0^\infty A(t)\,dt = \frac{N}{\beta}\left(\frac{\gamma}{\alpha+\gamma}\right)\ln\left(\frac{S_0}{S_\infty}\right). \tag{16}$$

Substituting (**??**) into (**??**), we obtain

$$S_\infty - N = -(\alpha+\kappa)\frac{N}{\beta}\left(\frac{\gamma}{\alpha+\gamma}\right)\ln\left(\frac{S_0}{S_\infty}\right). \tag{17}$$

Again rearranging the terms, we conclude that

$$\ln\left(\frac{S_0}{S_\infty}\right) - R_0\left(1-\frac{S_\infty}{N}\right) = 0, \tag{18}$$

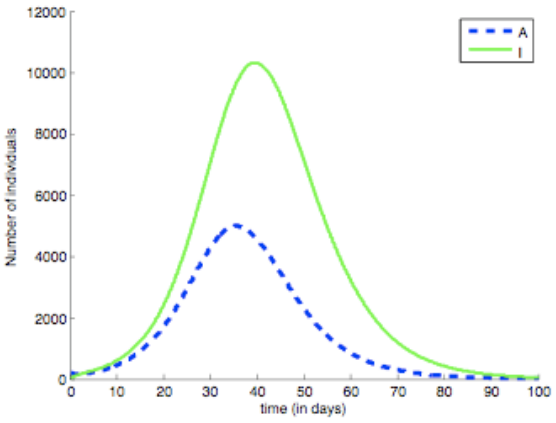
where

$$R_0 = \frac{\beta}{\alpha+\kappa}\left(\frac{\alpha+\gamma}{\gamma}\right) = \frac{\beta}{\alpha+\kappa} + \frac{\beta}{\alpha+\kappa}\frac{\alpha}{\gamma}. \tag{19}$$

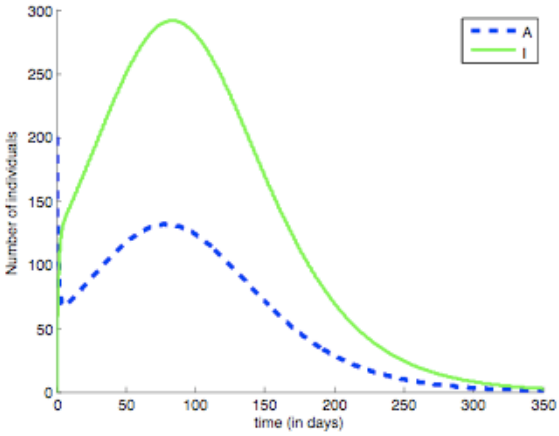
In the situation where asymptomatics must eventually show symptoms, the rate κ equals zero. The final size relation remains the same, with R_0 reducing to $\frac{\beta}{\alpha} + \frac{\beta}{\alpha}\frac{\alpha}{\gamma}$.

2.2 Simulations

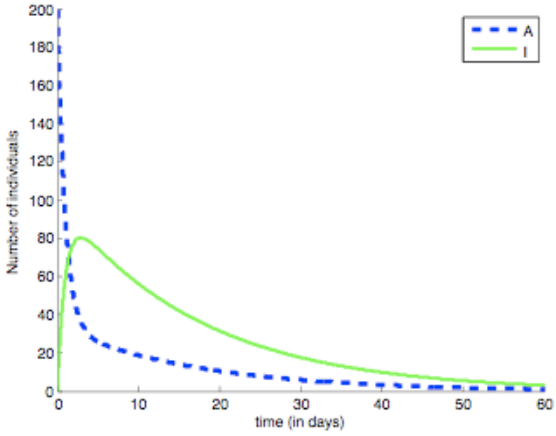
By looking at the final size relation (18)-(19), we notice that if κ increases, the duration and size of the outbreak decreases. Increasing κ reduces the infectious period of the asymptomatic individuals. Since asymptomatics move to the recovery compartment at a faster rate, susceptible individuals have fewer contacts with infectious individuals. This is also supported by the simulations.



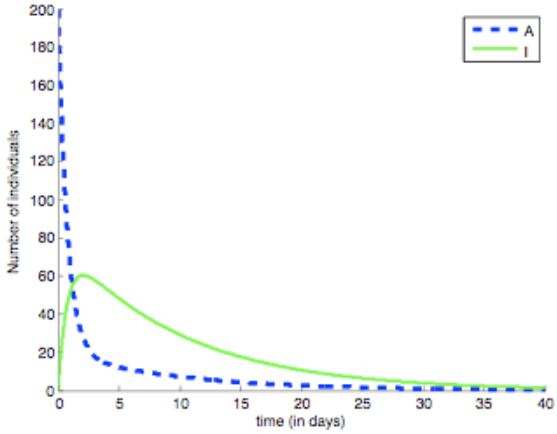
(a) SAIR with $k = 0$



(b) SAIR with $k = \frac{1}{3}$



(c) SAIR with $k = \frac{2}{3}$



(d) SAIR with $k = 1$

Figure 2: SAIR with varying k values **??,??,??,??**

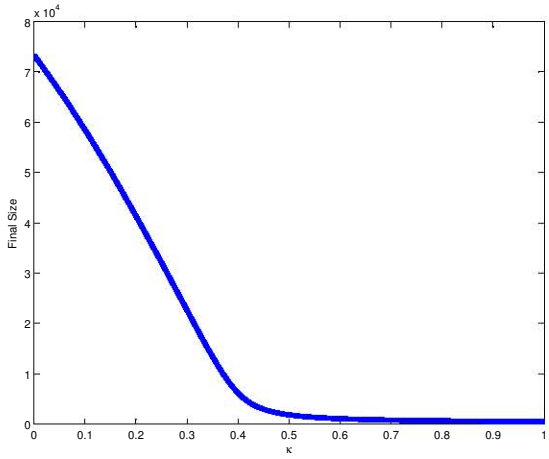


Figure 3: Final Size as a function of κ

2.3 Optimal Control Applied to Single-Strain SAIR Model

Control theory can assist in determining how to produce maximum performance or minimal cost [**?**]. First, we determined key variables that impact the outcome of a particular aspect of a problem. The theory then allowed us to obtain functions describing these variables, which produce the best outcome within certain reasonable constraints. Optimal control can be used to make public health policies in order to minimize the costs of an epidemic including illnesses, deaths and financial loss. In our model, we used control strategies to minimize the magnitude of the influenza outbreak. We have introduced treatment by controlling the rate at which individuals move from a symptomatic class to its corresponding recovered class. This objective in control theory is viewed as the "case finding". Case finding refers to pinpointing an infected individual and incorporating some type of intervention in order to produce a faster recovery [**?**]. In our model, the case finding will be a treatment. Also, control was applied to the transmission rates in order to decrease the number of infected individuals. This goal for reduction in infections is referred to as "case holding", where activities and techniques are seen as an effort to avoid contracting the infection [**?**]. In our model, the case holding is social distancing, such as closing schools or public events.

The H1N1 outbreak during the spring of 2009 brought up the question of what the best means are for controlling the size of such an epidemic. We address this question by applying two time-dependent controls to the SAIR model for H1N1 influenza. The first control, $u_1(t)$ is applied to the transmission rate β and represents the effort placed in social distancing. The second control, $u_2(t)$, represents effort placed in treatment of H1N1 infected individuals. These control functions are required to be a bounded, Lebesgue integrable on the interval $[0, t_f]$, where t_f is the duration of time for which we apply our controls. Since we only treat symptomatic individuals, we introduce

this term by adding $\gamma_T u_2(t)$, where γ_T is the additional recovery rate of an H1N1 infected individual undergoing treatment (i.e. $\gamma + \gamma_T$ = recovery rate with treatment), to the symptomatic equation ($\frac{dI}{dt}$). If $u_1(t)$ and $u_2(t)$ are equal to one, then full effort is being placed in social distancing and treatment, respectively, at time t. Conversely, if $u_1(t) = u_2(t) = 0$, then no effort is being placed in either control at time t.

We rewrite the state equations (**??**) to include the two control functions as follows:

$$\begin{aligned} \frac{dS}{dt} &= -\beta(1-u_1(t))S\left(\frac{A+I}{N}\right) \\ \frac{dA}{dt} &= \beta(1-u_1(t))S\left(\frac{A+I}{N}\right) - (\alpha+\kappa)A \\ \frac{dI}{dt} &= \alpha A - (\gamma + \gamma_T u_2(t))I \\ \frac{dR}{dt} &= (\gamma + \gamma_T u_2(t))I + \kappa A \end{aligned} \tag{20}$$

The benefits of implementing control measures are substantial, however each control incurs some cost. Social distancing generates economic losses, while treatment requires the infrastructure for administering the treatment. Thus, we give each control a weight which balances the relative cost of each control with their benefit. The relative cost of the controls is modeled by a quadratic term $\frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2$, where B_i represents the weight constant for the control u_i (i=1,2). Although the relative cost of each control depends on the population under consideration, for our simulations we make the general assumption that social distancing is more costly than treatment. Applying optimal control theory to the SAIR model for H1N1 allows us to determine the control functions $u_1^*(t)$ and $u_2^*(t)$ that produce the optimal outcome in the epidemic. Since our goal is to minimize the total number of infections during the H1N1 outbreak by applying the controls, we achieve this goal mathematically by minimizing an objective functional J, which incorporates the two infectious classes A and I. The objective functional J, a function of the controls, is defined as follows:

$$J(u_1(t), u_2(t)) = \int_0^{t_f} \left(A + I + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2\right) dt \tag{21}$$

The objective of determining the optimal control function pair (u_1^*, u_2^*) is expressed mathematically by:

$$J(u_1^*, u_2^*) = min_\Omega J(u_1, u_2), \tag{22}$$

where

$$\Omega = \{(u_1(t), u_2(t)) \in L^1(0, t_f) \| 0 \leq u_1(t), u_2(t) \leq 1, t \in [0, t_f]\}, \tag{23}$$

subject to the state equations (**??**) for given initial conditions.
Existence of the optimal controls u_1^* and u_2^* in our model are standard results in optimal control theory [**?**]. The necessary conditions that the optimal pair must satisfy are derived from Pontryagin's Maximum Principal. We first convert our problem, expressed by (**??**)-(**??**) into the equivalent problem of minimizing the Hamiltonian H:

$$\begin{aligned} H &= A + I + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 \\ &+ \lambda_1\left(-\beta(1-u_1(t))S\left(\frac{A+I}{N}\right)\right) \\ &+ \lambda_2\left(\beta(1-u_1(t))S\left(\frac{A+I}{N}\right) - (\alpha+\kappa)\right) \\ &+ \lambda_3\left(\alpha A - (\gamma+\gamma_T u_2(t))I\right) \\ &+ \lambda_4\left((\gamma+\gamma_T u_2(t))I + \kappa A\right) \end{aligned}$$

Standard use of Pontryagin's Maximum Principle ([**?**]) leads to the following Theorem:

Theorem 2.3.1 *There exists an optimal pair $u_1^*(t), u_2^*(t)$ and corresponding solutions, S^*, A^*, I^*, and R^*, that minimizes $J(u_1(t), u_2(t))$ over Ω. The explicit optimal controls are connected to the existence of continuous specific functions $\lambda_i(t)$, namely, the solutions of the following system (called the adjoint system) of differential equations:*

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1\beta(1-u_1(t))\left(\frac{A+I}{N}\right) - \lambda_2\beta(1-u_1(t))\left(\frac{A+I}{N}\right) \\ \frac{d\lambda_2}{dt} &= -1 + \lambda_1\beta(1-u_1(t))\frac{S}{N} - \lambda_2\left(\beta(1-u_1(t))\frac{S}{N} - (\alpha+\kappa)\right) - \lambda_3\alpha - \lambda_4\kappa \\ \frac{d\lambda_3}{dt} &= -1 + \lambda_1\beta(1-u_1(t))\frac{S}{N} - \lambda_2\beta(1-u_1(t))\frac{S}{N} + \lambda_3(\gamma+\gamma_T u_2(t)) - \lambda_4(\gamma+\gamma_T u_2(t)) \\ \frac{\lambda_4}{dt} &= 0 \end{aligned} \tag{24}$$

subject to the transversality conditions,

$$\lambda_i(t_f) = 0 \text{ for all } i = 1, 2, 3, 4. \tag{25}$$

Moreover, the following properties hold

$$\begin{aligned} u_1^* &= min\left(max\left(LB_1, \frac{1}{B_1}\left[\beta S\left(\frac{A+I}{N}\right)(\lambda_2-\lambda_1)\right]\right), UB_1\right) \\ u_2^* &= min\left(max\left(LB_2, \frac{1}{B_2}\left[\gamma_T I(\lambda_3-\lambda_4)\right]\right), UB_2\right), \end{aligned} \tag{26}$$

where LB_i is the desired lower bound for u_i and UB_i is the desired upper bound of u_i for $i = 1, 2$.

Proof The existence of an optimal control pair follows from Corollary 4.1 of [**?**] and the following two facts: the integrand of J is convex with respect to (u_1, u_2) and the state system is *Lipshitz* with respect to the state variables. The following relationships follow directly from the application of Pontryagin's Maximum Principle [**?**]:

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial S}, \ \frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial D}, \ \frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial R}, \ \frac{d\lambda_4(t)}{dt} = -\frac{\partial H}{\partial R},$$

with $\lambda_i(t_f) = 0$ for i=1,2, 3, and 4 evaluated at the optimal control pair and corresponding states. These evaluations naturally lead to the Adjoint System (**??**). The Hamiltonian H must be minimized with respect to the controls at the optimal control pair and so we differentiate H with respect to u_1 and u_2 on the set Ω. These computations lead to the following optimality conditions:

$$\frac{\partial H}{\partial u_1} = B_1 u_1 + (\lambda_1 - \lambda_2)\beta S\left(\frac{A+I}{N}\right) = 0 \text{ at } u_1 = u_1^*$$

$$\frac{\partial H}{\partial u_2} = B_2 u_2 + (-\lambda_3 + \lambda_4)\gamma_T I = 0 \text{ at } u_1 = u_1^*.$$

Solving for u_1^* and u_2^* gives

$$u_1^* = \frac{1}{B_1}\left[\beta S\left(\frac{A+I}{N}\right)(\lambda_2-\lambda_1)\right], u_2^* = \frac{1}{B_2}\left[\gamma_T I(\lambda_3-\lambda_4)\right],$$

Use of the bounds on $LB_i \leq u_i \leq UB_i$ for $i = 1, 2$ lead to the expressions in (**??**).

The optimality system consists of the State System (**??**) coupled with the Adjoint System (**??**), the initial conditions, transversality conditions (**??**), and the formulae in (**??**).

2.3.1 Numerical Results

We obtained our numerical results using the forward Euler method (See MATLAB code in Appendix **??**). Starting with an initial guess for the value of the controls at time $t = 0$, we solve the

state system with controls (**??**) using forward Euler. Next, the adjoint system is solved using the solutions from the state system and the transversality conditions (**??**) in backwards time. After updating the controls u_1 and u_2, the error between the old values of u_i $(i = 1, 2)$ and the updated value is calculated, and the process is repeated until the error is less than .001. When the process stops, the final values of u_1 and u_2 are numerical approximations of the optimal control pair (u_1^*, u_2^*). There are several means of updating the values of the controls u_1 and u_2. However, when the simpler approaches, such as taking the average of the current u_i and the u_i from the previous iteration, fail to converge, as with the SAIR model for H1N1, a convex combination with a weighted average can be used [**?**]. For our model, in order to avoid convergence problems, it was necessary to use a convex combination with a weighted average, which moves each iteration towards the current iteration [**?**].

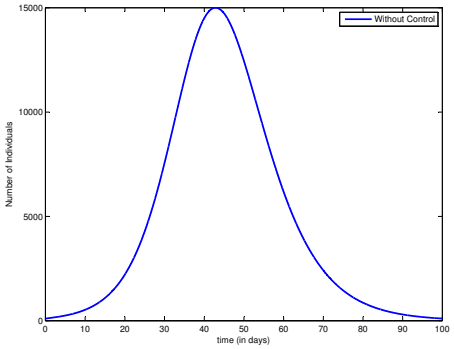
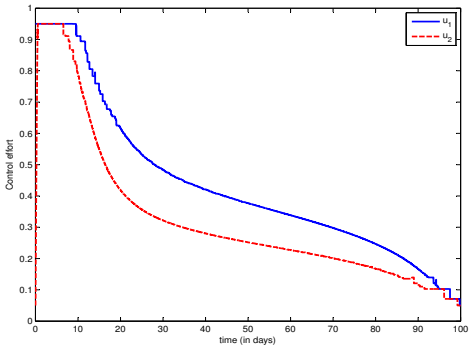
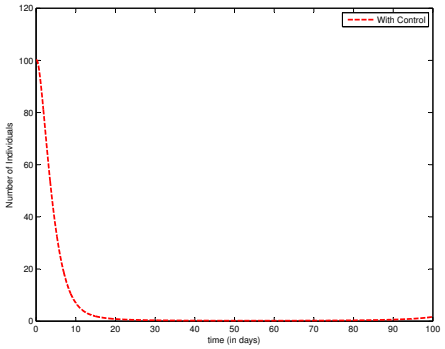


Figure 4: Total number of infected individuals $(A + I)$ at time t without control



(a) Optimal control functions u_1 and u_2



(b) Total number of infections $(A + I)$ With Control

Figure 5: SAIR optimal control results for $B_1 = 20$, $B_2 = 10$, and $A(0) = 100$ **??**,**??**

Applying social distancing and treatment controls to the model immediately suppresses the outbreak **??**. Our results showed that we should place full effort in both controls at the beginning of the outbreak. Although we chose social distancing to be the more costly of the two controls, figure **??** indicates that we should place more effort in this control rather than treatment throughout the duration of the outbreak. Since social distancing reduces the number of infectious contacts during the H1N1 epidemic, it is a more effective means of control than treatment. If we increase the relative cost of control u_1 by letting $B_1 = 50$, we observe that more effort should be placed in treatment. In this scenario, the higher relative cost of u_1 offsets some of the benefits of this control (see figure 6).

We then examined what effort is required if we only implement one control at a time, either social distancing or treatment, and how effectively each control reduces the outbreak. In figure 7(a) in which only social distancing is implemented, we still observed a significant reduction in the size of the outbreak, with full effort required only during the first twenty days. However, when the only control is treatment (figure 8(a)), even when full effort is made, the reduction in the outbreak is less substantial than with social distancing alone.

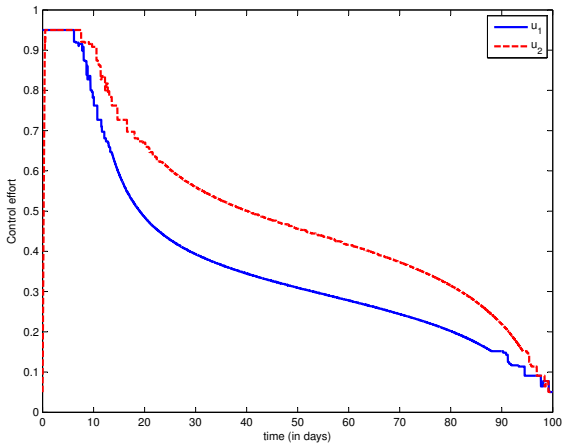
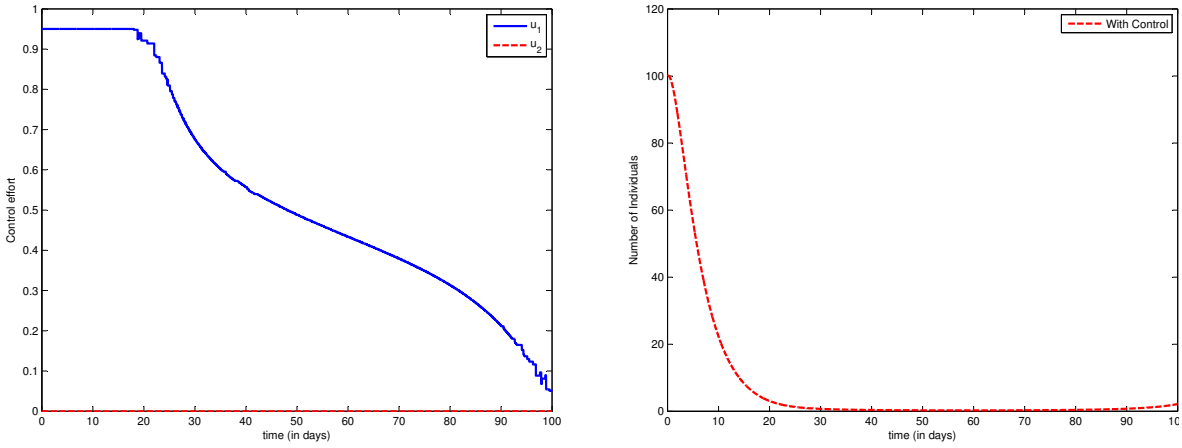


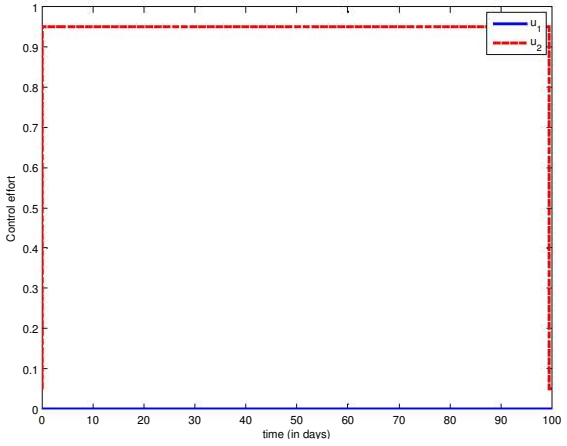
Figure 6: SAIR optimal control results for $B_1 = 50$, $B_2 = 10$, and $A(0) = 100$



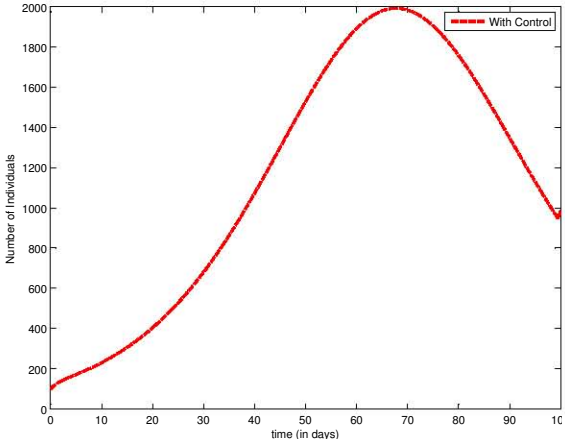
(a) Optimal control function u_1 with $u_2 = 0$

(b) Total number of infections $(A + I)$ With Control

Figure 7: SAIR optimal control results for $B_1 = 20$, $B_2 = 10$, and $A(0) = 100$ **??,??**



(a) Optimal control function u_2 with $u_1 = 0$



(b) Total number of infections $(A + I)$ With Control

Figure 8: SAIR optimal control results for $B_1 = 20$, $B_2 = 10$, and $A(0) = 100$ **??**,**??**

3 Two Strain Model

Initially, everyone is susceptible to both strains of influenza. Individuals can be infected with both the seasonal influenza and H1N1 virus in a single outbreak, but never at the same time. Another assumption is that once an individual has been infected with and recovered from either H1N1 or seasonal flu, an individual can still be infected with the influenza they haven't contracted yet. Also, it will be assumed that only the seasonal vaccine is available and there is no cross-immunity. Once an individual has been infected with a type of influenza, he/she can either die or go to a recovered class where he/she is considered immune to the infection of that type. There are no vital dynamics, meaning natural births and deaths are not being considered. The seasonal vaccine is assumed to be 100% effective and the recovery rates for the two influenzas are assumed to be different. People who exhibit flu-like symptoms during a flu outbreak, who do not have influenza, are considered negligible because the outbreak is taking place during the flu season. Asymptomatics (A), as well as symptomatics (I), are being considered in the model because they play a vital role in disease transmission and vaccination.

In this single outbreak model, it is assumed that all individuals that become infected will eventually show symptoms. This is the reason for having those in the A classes going into the I classes. Only individuals who do not show symptoms will be given the seasonal vaccine(V_S, V_{A1}, V_{A2}). For example, those who have no previous infections (S) or those who have not shown flu-like symptoms during the outbreak $(A_1$,$A_2)$ will be eligible for vaccination. People who have been vaccinated are not considered immune to the H1N1 virus in the model, and therefore can be infected with H1N1 $(A_{1,2}, I_{1,2}, A_{2,1}, I_{2,1}, A_2^*, I_2^*)$. Once in the symptomatic classes, an individual will either die or go to

the protected class (P), where they are considered protected against re-infection from both strains. The area that represents waste in the diagram is when a person who has asymptomatic seasonal flu gets a seasonal vaccine. These people are still infectious and will eventually show symptoms.

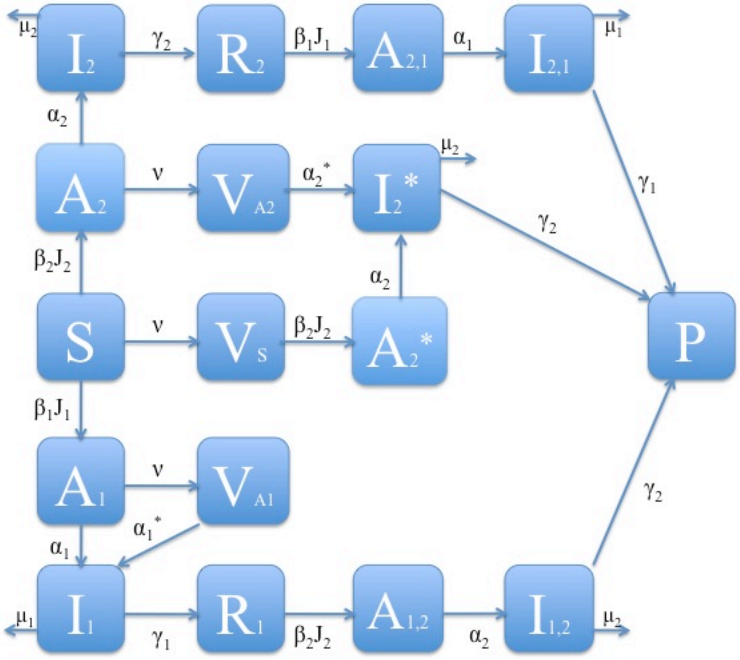


Figure 9: Diagram of the Compartmental Model, where 1 =Seasonal and 2 =H1N1

Table 2: List of Classes and Their Meaning, where 1 = Seasonal and 2 = H1N1

Class	Meaning
S	Susceptible
A_1	Infected with Seasonal- Not Showing Symptoms (No Previous Infection)
I_1	Infected with Seasonal - Showing Symptoms (No Previous Infection)
R_1	Recovered from Seasonal
A_2	Infected with H1N1- Not Showing Symptoms (No Previous Infection)
I_2	Infected with H1N1 - Showing Symptoms (No Previous Infection)
R_2	Recovered from H1N1
V_s	Vaccine Given to Susceptible
V_{A1}	Vaccine Given to Asymptomatic with Seasonal Flu
V_{A2}	Vaccine Given to Asymptomatic with H1N1 Virus
A_2^*	Infected with H1N1 after Receiving Vaccine- Not Showing Symptoms
I_2^*	Infected with H1N1 after Receiving Vaccine- Showing Symptoms
$A_{1,2}$	Previously Infected with Seasonal, now Infected with H1N1- Not Showing Symptoms
$A_{2,1}$	Previously Infected with H1N1, now Infected with Seasonal- Not Showing Symptoms
$I_{1,2}$	Previously Infected with Seasonal, now Infected with H1N1- Showing Symptoms
$I_{2,1}$	Previously Infected with H1N1, now Infected with Seasonal- Showing Symptoms
P	Protected Against Seasonal and H1N1

Table 3: List of Parameters and their Values

Type	Parameters	Description	Value	Reference
Seasonal	β_1	Transmission Rate	0.2167	Estimation
	α_1	Rate of Progression to Symptomatic	$\frac{1}{2}$	Estimation
	α_1^*	Rate to Symptomatic Infection after Vaccination	.5	Estimation
	γ_1	Recovery Rate	$\frac{1}{5}$	[?]
	μ_1	Death Rate	10^{-6}	Estimation
	ν	Vaccination Rate	.01	Estimation
	R_o	Basic Reproductive Number	1.3	[?]
H1N1	β_2	Transmission Rate	0.2793	Estimation
	α_2	Rate of Progression to Symptomatic	$\frac{1}{2}$	[?]
	α_2^*	Rate to Symptomatic Infection after Vaccination	.5	Estimation
	γ_2	Recovery Rate	$\frac{1}{33} : \frac{1}{100}$	[?]
	μ_2	Death Rate	10^{-6}	Estimation
	R_o	Basic Reproductive Number	1.8	[?]

3.1 Differential Equations

$$\frac{dS}{dt} = -\beta_1 S J_1 - \beta_2 S J_2 - \nu S \tag{27}$$
$$\frac{dA_1}{dt} = \beta_1 S J_1 - (\alpha_1 + \nu) A_1 \tag{28}$$
$$\frac{dI_1}{dt} = \alpha_1 A_1 + \alpha_1^* V_{A1} - (\gamma_1 + \mu_1) I_1 \tag{29}$$
$$\frac{dR_1}{dt} = \gamma_1 I_1 - \beta_2 R_1 J_2 \tag{30}$$
$$\frac{dA_{1,2}}{dt} = \beta_2 R_1 J_2 - \alpha_2 A_{1,2} \tag{31}$$
$$\frac{dI_{1,2}}{dt} = \alpha_2 A_{1,2} - (\gamma_2 + \mu_2) I_{1,2} \tag{32}$$
$$\frac{dA_2}{dt} = \beta_2 S J_2 - (\alpha_2 + \nu) A_2 \tag{33}$$
$$\frac{dI_2}{dt} = \alpha_2 A_2 - (\gamma_2 + \mu_2) I_2 \tag{34}$$
$$\frac{dR_2}{dt} = \gamma_2 I_2 - \beta_1 R_2 J_1 \tag{35}$$
$$\frac{dA_{2,1}}{dt} = \beta_1 R_2 J_1 - \alpha_1 A_{2,1} \tag{36}$$
$$\frac{dI_{2,1}}{dt} = \alpha_1 A_{2,1} - (\gamma_1 + \mu_1) I_{2,1} \tag{37}$$
$$\frac{dV_{A2}}{dt} = \nu A_2 - \alpha_2^* V_{A2} \tag{38}$$
$$\frac{dI_2^*}{dt} = \alpha_2^* V_{A2} + \alpha_2 A_2^* - (\gamma_2 + \mu_2) I_2^* \tag{39}$$
$$\frac{dV_S}{dt} = \nu S - \beta_2 V_S J_2 \tag{40}$$
$$\frac{dA_2^*}{dt} = \beta_2 V_S J_2 - \alpha_2 A_2^* \tag{41}$$
$$\frac{dV_{A1}}{dt} = \nu A_1 - \alpha_1^* V_{A1} \tag{42}$$
$$\frac{dP}{dt} = \gamma_2 (I_2^* + I_{1,2}) + \gamma_1 I_{2,1} \tag{43}$$

where

$$J_1 = \frac{I_1 + A_1 + A_{2,1} + I_{2,1} + V_{A_1}}{N}$$
$$J_2 = \frac{I_2 + A_2 + A_{1,2} + I_{1,2} + I_2^* + A_2^* + V_{A2}}{N}$$

4 Analysis

4.1 Calculating R_0

$$
\mathcal{F} = \begin{bmatrix} \beta_1 S J_1 \\ \beta_1 R_1 J_1 \\ \beta_2 S J_2 \\ \beta_2 R_1 J_2 \\ \beta_2 V_s J_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad
\mathcal{V} = \begin{bmatrix} (\alpha_1 + \nu) A_1 \\ \alpha_1 A_{2,1} \\ (\alpha_2 + \nu) A_2 \\ \alpha_2 A_{1,2} \\ \alpha_2 A_2^* \\ -\alpha_1 A_1 - \alpha_1^* V_{A1} + (\alpha_1 + \mu_1) I_1 \\ -\alpha_1 A_{2,1} + (\alpha_1 + \mu_1) I_{2,1} \\ -\nu A_1 + \alpha_1^* V_{A1} \\ -\alpha_2 A_2 + (\gamma_2 + \mu_2) I_2 \\ -\alpha_2 A_{1,2} + (\alpha_2 + \mu_2) I_{1,2} \\ -\nu A_2 + \alpha_2^* V_2 \\ -\alpha_2^* V_{A2} - \alpha_2 A_2^* + (\gamma_2 + \mu_2) I_2^* \end{bmatrix}
$$

Taking the Jacobian of both of these matrices, $\mathcal{F}, \mathcal{V}$, we obtain matrices F and V, respectively

$$
F = \begin{bmatrix}
\beta_1 & \beta_1 & 0 & 0 & 0 & \beta_1 & \beta_1 & \beta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_2 & \beta_2 & \beta_2 & 0 & 0 & 0 & \beta_2 & \beta_2 & \beta_2 & \beta_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
V = \begin{bmatrix}
\alpha_1 + \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_2 + \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha_1 & 0 & 0 & 0 & 0 & \gamma_1 + \mu_1 & 0 & -\alpha_1^* & 0 & 0 & 0 & 0 \\
0 & -\alpha_1 & 0 & 0 & 0 & 0 & \gamma_1 + \mu_1 & 0 & 0 & 0 & 0 & 0 \\
-\nu & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1^* & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 & \gamma_2 + \mu_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 & \gamma_2 + \mu_2 & 0 & 0 \\
0 & 0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2^* & 0 \\
0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 & -\alpha_2^* & \gamma_2 + \mu_2
\end{bmatrix}
$$

Using the inverse of V(see Appendix A), we multiply it by F, where $A = \dfrac{\beta_1}{\gamma_1 + \mu_1}$, $B = \dfrac{\beta_2}{\gamma_2 + \mu_2}$, and $C = \dfrac{\beta_2}{\alpha_2 + \nu} + \dfrac{\beta_2}{\gamma_2 + \mu_2}\dfrac{\alpha_2}{\alpha_2 + \nu} + \dfrac{\beta_2 \nu}{(\alpha_2 + \nu)\alpha_2^*} + \dfrac{\beta_2}{\gamma_2 + \mu_2}\dfrac{\nu}{\alpha_2 + \nu}$ to acquire

$$FV^{-1} = \begin{bmatrix} \frac{\beta_1}{\alpha_1+\nu} + A + \frac{\beta_1 \nu}{(\alpha_1+\nu)\alpha_1^*} & \frac{\beta_1}{\alpha_1} + A & 0 & 0 & 0 & A & A & A + \frac{\beta_1}{\alpha_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & \frac{\beta_2}{\alpha_2} + B & \frac{\beta_2}{\alpha_2} + B & 0 & 0 & 0 & B & B & \frac{\beta_2}{\alpha_2^*} + B & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since FV^{-1} is an upper triangular matrix, the eigenvalues are equal to the values of the diagonal. So, the basic controlled reproductive number is

$$R_c = max\{\lambda_1, \lambda_2\}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\beta_1}{\alpha_1 + \nu} + \frac{\beta_1}{\alpha_1 + \nu}\frac{\alpha_1}{\gamma_1 + \mu_1} + \frac{\beta_1}{\alpha_1 + \nu}\frac{\nu}{\alpha_1^*}\frac{\alpha_1^*}{\gamma_1 + \mu_1} + \frac{\beta_1}{\alpha_1 + \nu}\frac{\nu}{\alpha_1^*} \\ \lambda_2 &= \frac{\beta_2}{\alpha_2 + \nu} + \frac{\beta_2}{\alpha_2 + \nu}\frac{\alpha_2}{\gamma_2 + \mu_2} + \frac{\beta_2}{\alpha_2 + \nu}\frac{\nu}{\alpha_2^*}\frac{\alpha_2^*}{\gamma_2 + \mu_2} + \frac{\beta_2}{\alpha_2 + \nu}\frac{\nu}{\alpha_2^*} \end{aligned} \tag{37}$$

The terms in each eigenvalue represent the possible paths an individual will take through the different classes in our model. They also only account for the cases when an individual is infected with influenza for the first time during the outbreak. In λ_1, which represents the basic reproductive number for seasonal flu, an individual can either 1. become infected with seasonal and not show symptoms (A_1), 2. become infected with seasonal and take two days to show symptoms ($A_1 I_1$), or 3. become infected with seasonal and not show symptoms, get vaccinated for seasonal flu, and then develop symptoms ($A_1 V_{A1} I_1$). In λ_2, which represents H1N1 virus, an individual can either 1. become infected with H1N1 and not show symptoms (A_2), 2. become infected with H1N1 and take two days to show symptoms ($A_2 I_2$), 3. become infected with H1N1 without showing symptoms and then get vaccinated for seasonal flu ($A_2 V_{A2}$), or 4. become infected with H1N1 without showing

symptoms, get vaccinated for seasonal flu, then develop symptoms for H1N1 ($A_2V_{A2}I_2^*$).

In the absence of control, no vaccination, R_c reduces to the basic reproductive number, R_0.

Thus, the basic reproductive numbers for seasonal influenza and H1N1 are

$$\begin{aligned} R_{10} &= \frac{\beta_1}{\alpha_1} + \frac{\beta_1}{\alpha_1}\frac{\alpha_1}{\gamma_1 + \mu_1} \\ R_{20} &= \frac{\beta_2}{\alpha_2} + \frac{\beta_2}{\alpha_2}\frac{\alpha_2}{\gamma_2 + \mu_2} \end{aligned}$$

respectively. Thus, $R_0 = max\{R_{10}, R_{20}\}$.
The SAIR model in section **??** with $\kappa = 0$ is a special case of our more complex two-strain model with vaccination. Setting the death rate, vaccination rate, and seasonal flu infection rates equal to zero ($\mu = \nu = \beta_1 = 0$) to recreate the setting of the SAIR model for seasonal influenza when $\kappa = 0$, we observe that $R_0 = max\{R_{20}, 0\} = R_{20}$ is consistent with the basic reproductive number derived in section **??**

$$\begin{aligned} R_{20} &= \frac{\beta_2}{\alpha_2} + \frac{\beta_2}{\alpha_2}\frac{\alpha_2}{\gamma_2 + \mu_2} \\ &= \frac{\beta}{\alpha} + \frac{\beta}{\alpha}\frac{\alpha}{\gamma}. \end{aligned}$$

4.2 Simulations of Two-Strain Model with and without Vaccination

Numerical simulations allow us to evaluate the effects of vaccination on the duration of an outbreak, as well as the total morbidity at the end of the outbreak of seasonal and H1N1 influenza. In the presence of vaccination, we can also use simulations to count the total number of wasted vaccines during an outbreak for different population sizes and initial conditions. In tables Table 4 and Table 5, we summarize some results for a population of 100,000 individuals.

In figures 10 and 11, we compared the outbreaks of seasonal and H1N1 influenza for different initial conditions, dividing the infectious individuals according to the different infectious states - A_1, I_1, A_{21}, I_{21} for seasonal influenza infections, and A_2, I_2, A_{12}, I_{12} for H1N1 infections. The graphs demonstrate a much larger peak in the H1N1 outbreak than for seasonal influenza; however, the duration of H1N1 is much shorter. We observed different dynamics in the growth of the A_1 class in figure 10(a) and 11(a). When there are more asymptomatic seasonal infections than asymptomatic H1N1 infections at the beginning of the outbreak, we noticed fluctuations in the growth of A_1. Initially, A_1 increases then begins to decrease as individuals recover from seasonal influenza. Then, as individuals recover from I_2, they are now susceptible to a secondary infection from a seasonal virus. Because the H1N1 outbreak is large, the number of secondary seasonal infections is greater

than the number of primary seasonal infections at the beginning of the outbreak. So, as members of the population enter the A_{21} class, they begin to infect members of class S, resulting in another visible increase in the A_1 class.

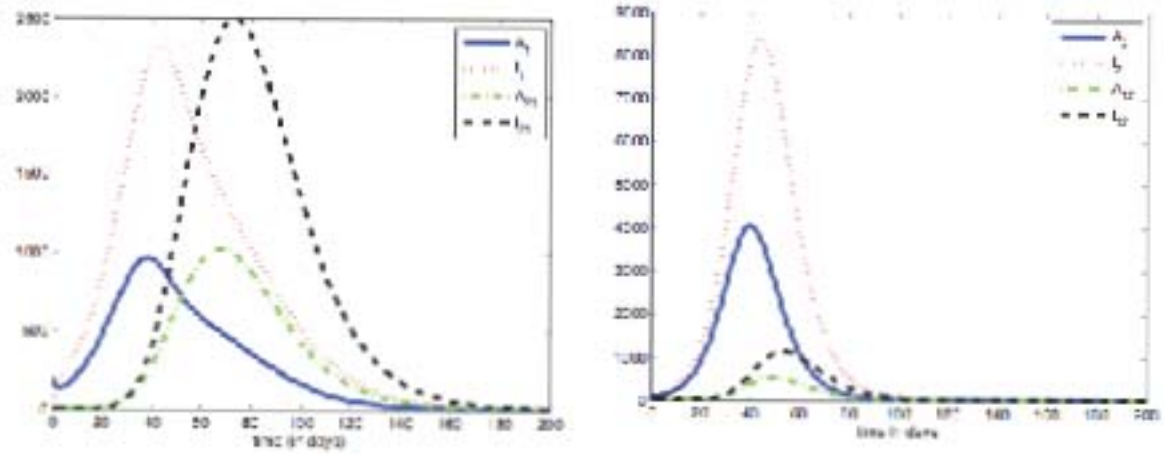
Table 4: Results for varying initial conditions with vaccination

	$A_1(0) = 200$ $A_2(0) = 100$	$A_1(0) = 100$ $A_2(0) = 200$	$A_1(0) = 100$ $A_2(0) = 0$	$A_1(0) = 0$ $A_2(0) = 100$
Total Wasted Vaccines	110	55	97	0
Total Successful Vaccinations	48,114	46,812	95,038	50,456
Total H1N1's Vaccinated	906	987	0	971
Total Unvaccinated	50,869	52,145	4,864	48,572
Total Infected	74,801	74,101	4,857	73,192
Total H1N1 Infections	72,577	72,800	0	73,192

Table 5: Results for varying initial conditions without vaccination

	$A_1(0) = 200$ $A_2(0) = 100$	$A_1(0) = 100$ $A_2(0) = 200$	$A_1(0) = 100$ $A_2(0) = 0$	$A_1(0) = 0$ $A_2(0) = 100$
Total Infected	87,256	88,047	59,367	73,192
Total H1N1 Infections	69,602	71,202	0	73,192

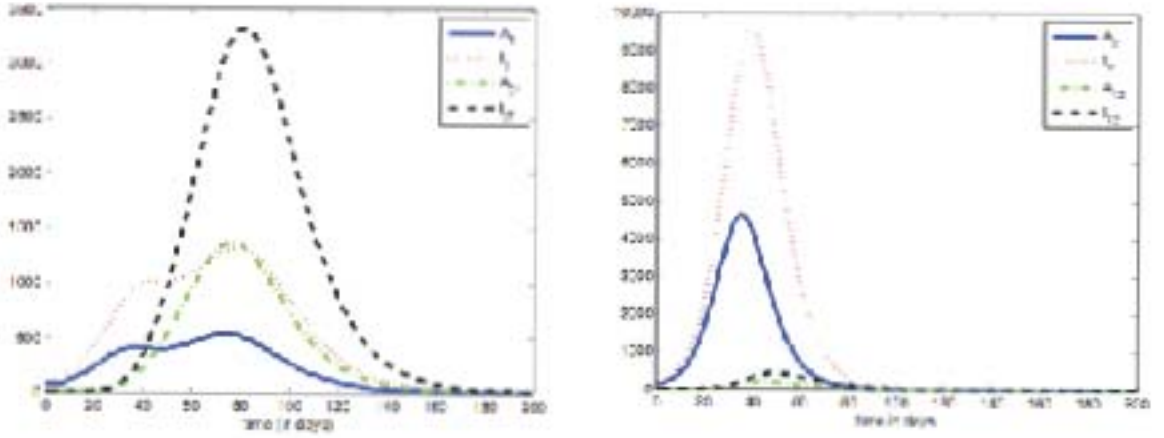
Vaccination significantly changes the dynamics of the seasonal influenza infections. In our model without vaccination, more individuals contract secondary (I_{21}) seasonal infections than primary (I_1) seasonal infections. However, when vaccination is introduced, we observed a higher peak in the primary seasonal infections than in the secondary seasonal infections. Since our model assumes no co-infection of influenza virus strains, using the seasonal vaccine has a negative effect in terms of the H1N1 outbreak. Allowing more individuals to become infected with seasonal influenza, by withholding vaccination, reduces the number of people susceptible to H1N1. Thus, we observe a slight increase in the number of H1N1 infections (see Tables 4 and 5) when vaccination is present in the model.



(a) Seasonal Infectious classes

(b) H1N1 Infectious classes

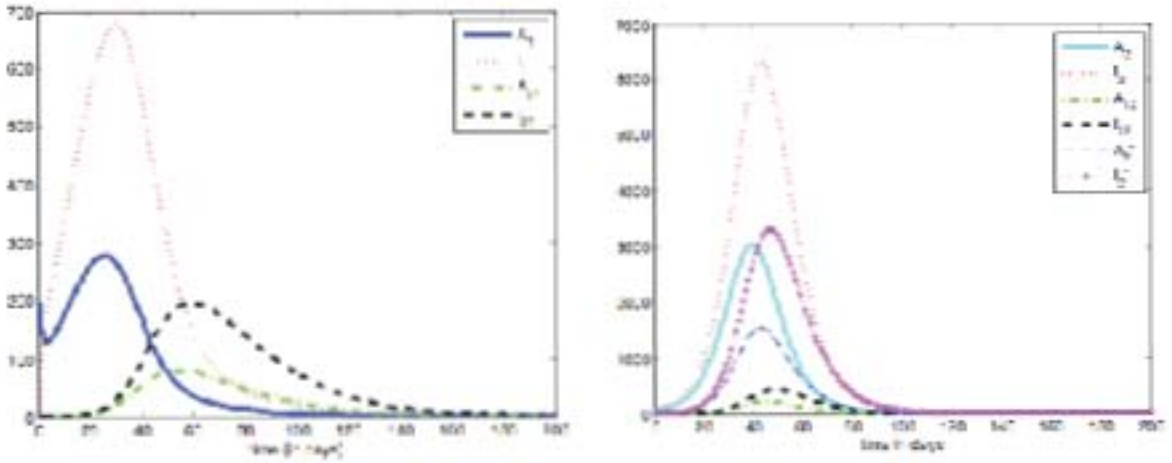
Figure 10: Unvaccinated Infectious Classes when $A_1(0) = 200$ and $A_2(0) = 100$



(a) Seasonal Infectious classes

(b) H1N1 Infectious classes

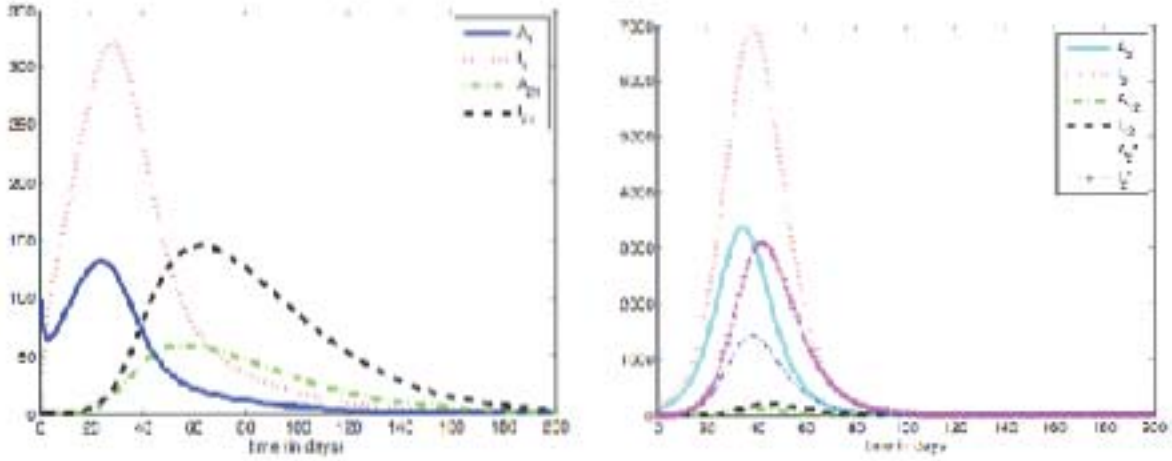
Figure 11: Unvaccinated Infectious Classes when $A_1(0) = 100$ and $A_2(0) = 200$



(a) Seasonal Infectious classes

(b) H1N1 Infectious classes

Figure 12: Vaccinated Infectious Classes when $A_1(0) = 200$ and $A_2(0) = 100$

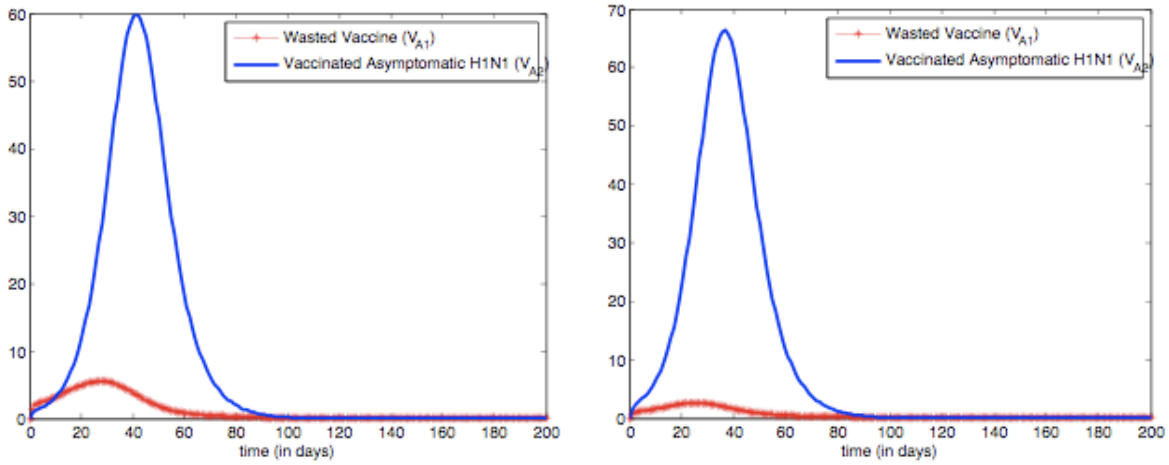


(a) Seasonal Infectious classes

(b) H1N1 Infectious classes

Figure 13: Vaccinated Infectious Classes when $A_1(0) = 100$ and $A_2(0) = 200$

In addition to determining the effectiveness of a particular strategy, when a resource is limited we also want to evaluate the wastefulness of the strategy that decides the allocation of these resources. Under our vaccination strategy, we only vaccinate those who have never shown symptoms of an influenza infection during the current outbreak or who have received a vaccine for the current flu season. There will always be wasted vaccines in this strategy when seasonal influenza is present because we will vaccinate a portion of individuals who are already infected with seasonal influenza and simply have not yet shown symptoms (A_1). To count the total number of vaccines wasted, we calculate the total number of people who have passed through the vaccinated asymptomatic seasonal compartment (V_{A1}) during the outbreak. Since the vaccination rate (.01) is low in comparison to the rate of becoming symptomatic (0.5), we observe low numbers of wasted vaccine (Table 5). Our strategy also allows us to vaccinate those who are asymptomatic with H1N1 and have had no previous influenza infections. These vaccinations are successful vaccinations since once a person recovers from H1N1, they become susceptible to seasonal influenza if they have not received a seasonal vaccine. By counting the number of individuals who have traveled through V_{A2} during the outbreak, we discovered that our strategy successfully vaccinates many more primary H1N1 infected individuals (A_2) than the number of vaccines it wastes on primary asymptomatic seasonal influenza infected individuals (see figure 14 and Table 4). However, because we cannot distinguish between seasonal influenza and H1N1 influenza symptoms, our vaccination strategy will also fail to vaccinate those who have not been infected with seasonal flu and who either have symptomatic H1N1 (I_2) or have recovered from H1N1 (R_2). Hence, our strategy may be useful in societies that have limited resources to administer efficient tests for H1N1 to their patients and cannot afford to waste the available vaccines.



(a) V_{A1} (Wasted Vaccine) and V_{A2}

(b) V_{A1} (Wasted Vaccine) and V_{A2}

Figure 14: In **??** $A_1(0) = 200$ and $A_2(0) = 100$, in **??** $A_1(0) = 100$ and $A_2(0) = 200$

5 Optimal Control Theory Applied to Two-Strain Model

In section **??**, we introduced controls to the single-strain SAIR model for H1N1. We explored the same controls, social distancing (u_1) and treatment (u_2), in our two-strain influenza model. Since social distancing affects a population as a whole, we added this control to the transmission rate of seasonal influenza β_1 in addition to β_2. However, we chose to only provide treatment for those symptomatic with H1N1. This assumes that patients are tested for the H1N1 subtype of influenza before treatment is administered.

The state system equations for our 17 differential equations will be:

$$\frac{dS}{dt} = -\beta_1(1-u_1)SJ_1 - \beta_2(1-u_1)SJ_2 - \nu S \tag{34}$$

$$\frac{dA_1}{dt} = \beta_1(1-u_1)SJ_1 - (\alpha_1+\nu)A_1 \tag{35}$$

$$\frac{dI_1}{dt} = \alpha_1 A_1 + \alpha_1^* V_{A1} - (\gamma_1+\mu_1)I_1 \tag{36}$$

$$\frac{dR_1}{dt} = \gamma_1 I_1 - \beta_2(1-u_1)R_1J_2 \tag{37}$$

$$\frac{dA_{1,2}}{dt} = \beta_2(1-u_1)R_1J_2 - \alpha_2 A_{1,2} \tag{38}$$

$$\frac{dI_{1,2}}{dt} = \alpha_2 A_{1,2} - (\gamma_2(1+u_2)+\mu_2)I_{1,2} \tag{39}$$

$$\frac{dA_2}{dt} = \beta_2(1-u_1)SJ_2 - (\alpha_2+\nu)A_2 \tag{40}$$

$$\frac{dI_2}{dt} = \alpha_2 A_2 - (\gamma_2(1+u_2)+\mu_2)I_2 \tag{41}$$

$$\frac{dR_2}{dt} = \gamma_2(1+u_2)I_2 - \beta_1(1-u_1)R_2J_1 \tag{42}$$

$$\frac{dA_{2,1}}{dt} = \beta_1(1-u_1)R_2J_1 - \alpha_1 A_{2,1} \tag{43}$$

$$\frac{dI_{2,1}}{dt} = \alpha_1 A_{2,1} - (\gamma_1+\mu_1)I_{2,1} \tag{44}$$

$$\frac{dV_{A2}}{dt} = \nu A_2 - \alpha_2^* V_{A2} \tag{45}$$

$$\frac{dI_2^*}{dt} = \alpha_2^* V_{A2} + \alpha_2 A_2^* - (\gamma_2(1+u_2)+\mu_2)I_2^* \tag{46}$$

$$\frac{dV_S}{dt} = \nu S - \beta_2(1-u_1)V_S J_2 \tag{47}$$

$$\frac{dA_2^*}{dt} = \beta_2(1-u_1)V_S J_2 - \alpha_2 A_2^* \tag{48}$$

$$\frac{dV_{A1}}{dt} = \nu A_1 - \alpha_1^* V_{A1} \quad (49)$$

$$\frac{dP}{dt} = \gamma_2(1+u_2)(I_2^* + I_{1,2}) + \gamma_1 I_{2,1} \quad (50)$$

Our goal is to reduce the number of seasonal and H1N1 infections and also increase the number of recovered individuals. We used the same technique as with the SAIR model for deriving the optimal control pair (u_1^*, u_2^*). The objective functional to be minimized is

$$\mathbb{J}(u_1, u_2) = \int_0^{t_f} [J_1(t)N + J_2(t)N + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t)]dt$$

where u_1 is controlling β_1 and β_2 and u_2 is controlling γ_2. From Pontryagin's Maximum Principle, we find the optimal controls by minimizing a Hamiltonian, H, where

$$H = J_1(t)N + J_2(t)N + \tfrac{B_1}{2}u_1^2 + \tfrac{B_2}{2}u_2^2 + \sum_{i=1}^{17}\lambda_i g_i$$

Also, from using Pontryagin's Maximum Principle, we gather that

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S}, \lambda_1(t_f) = 0$$
$$\cdots$$
$$\frac{d\lambda_{17}}{dt} = -\frac{\partial H}{\partial P}, \lambda_{17}(t_f) = 0$$

From this expression, we obtain the adjoint system:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1(\beta_1(1-u_1^*)J_1 + \beta_2(1-u_1^*)J_2 + \nu) - \lambda_2(\beta_1(1-u_1^*)J_1) - \lambda_7(\beta_2(1-u_1^*)J_2) - \lambda_{14}(\nu) \\
\frac{d\lambda_2}{dt} &= -1 + \lambda_1(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_2(\beta_1(1-u_1^*)\frac{S}{N} - (\alpha_1+\nu)) - \lambda_3(\alpha_1) + \lambda_9(\beta_1(1-u_1^*)\frac{R_2}{N}) \\
&\quad -\lambda_{10}(\beta_1(1-u_1^*)\frac{R_2}{N}) - \lambda_{16}(\nu) \\
\frac{d\lambda_3}{dt} &= -1 + \lambda_1(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_2(\beta_1(1-u_1^*)\frac{S}{N}) + \lambda_3(\gamma_1+\mu_1) - \lambda_4(\gamma_1) + \lambda_9(\beta_1(1-u_1^*)\frac{R_2}{N}) \\
&\quad -\lambda_{10}(\beta_1(1-u_1^*)\frac{R_2}{N}) \\
\frac{d\lambda_4}{dt} &= \lambda_4(\beta_2(1-u_1^*)J_2) - \lambda_5(\beta_2(1-u_1^*)J_2) \\
\frac{d\lambda_5}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N} - \alpha_2) - \lambda_6(\alpha_2) \\
&\quad -\lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N})
\end{aligned}$$

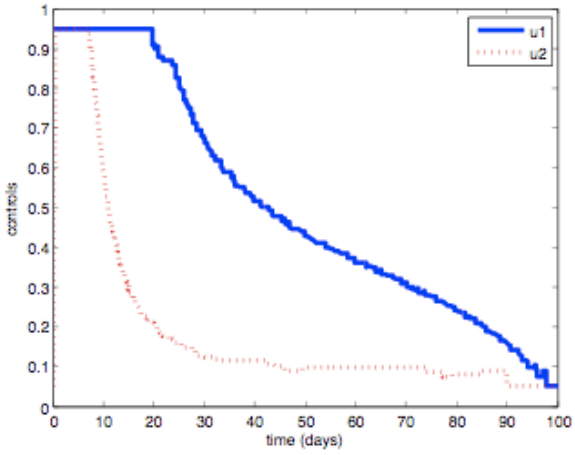
$$
\begin{aligned}
\frac{d\lambda_6}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) + \lambda_6(\gamma_2(1+u_2^*)+\mu_2) \\
&\quad -\lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{17}(\gamma_2(1+u_2^*)) \\
\frac{d\lambda_7}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_7(\beta_2(1-u_1^*)\frac{S}{N} + (\alpha_2+\nu)) \\
&\quad -\lambda_8(\alpha_2) - \lambda_{12}(\nu) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N}) \\
\frac{d\lambda_8}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) \\
&\quad +\lambda_8(\gamma_2(1+u_2^*)+\mu_2) - \lambda_9(\gamma_2(1+u_2^*)) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N}) \\
\frac{d\lambda_9}{dt} &= \lambda_9(\beta_1(1-u_1^*)J_1) - \lambda_{10}(\beta_1(1-u_1^*)J_1) \\
\frac{d\lambda_{10}}{dt} &= -1 + \lambda_1(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_2(\beta_1(1-u_1^*)\frac{S}{N}) + \lambda_9(\beta_1(1-u_1^*)\frac{R_2}{N}) - \lambda_{10}(\beta_1(1-u_1^*)\frac{R_2}{N} - \alpha_1) \\
&\quad -\lambda_{11}(\alpha_1) \\
\frac{d\lambda_{11}}{dt} &= -1 + \lambda_1(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_2(\beta_1(1-u_1^*)\frac{S}{N}) + \lambda_9(\beta_1(1-u_1^*)\frac{R_2}{N}) - \lambda_{10}(\beta_1(1-u_1^*)\frac{R_2}{N}) \\
&\quad +\lambda_{11}(\gamma_1+\mu_1) - \lambda_{17}(\gamma_1) \\
\frac{d\lambda_{12}}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) \\
&\quad +\lambda_{12}(\alpha_2^*) - \lambda_{13}(\alpha_2^*) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N}) \\
\frac{d\lambda_{13}}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) \\
&\quad +\lambda_{13}(\gamma_2(1+u_2^*)+\mu_2) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{17}(\gamma_2(1+u_2^*)) \\
\frac{d\lambda_{14}}{dt} &= -\lambda_{15}(\beta_2(1-u_1^*)J_2) + \lambda_{14}(\beta_2(1-u_1^*)J_2) \\
\frac{d\lambda_{15}}{dt} &= -1 + \lambda_1(\beta_2(1-u_1^*)\frac{S}{N}) + \lambda_4(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_5(\beta_2(1-u_1^*)\frac{R_1}{N}) - \lambda_7(\beta_2(1-u_1^*)\frac{S}{N}) \\
&\quad -\lambda_{13}(\alpha_2) + \lambda_{14}(\beta_2(1-u_1^*)\frac{V_S}{N}) - \lambda_{15}(\beta_2(1-u_1^*)\frac{V_S}{N} - \alpha_2) \\
\frac{d\lambda_{16}}{dt} &= -1 + \lambda_1(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_2(\beta_1(1-u_1^*)\frac{S}{N}) - \lambda_3(\alpha_1^*) + \lambda_9(\beta_1(1-u_1^*)\frac{R_2}{N}) - \lambda_{10}(\beta_1(1-u_1^*)\frac{R_2}{N}) \\
&\quad +\lambda_{16}(\alpha_1^*) \\
\frac{d\lambda_{17}}{dt} &= 0
\end{aligned}
$$

The optimal control pair (u_1^*, u_2^*) is defined by:

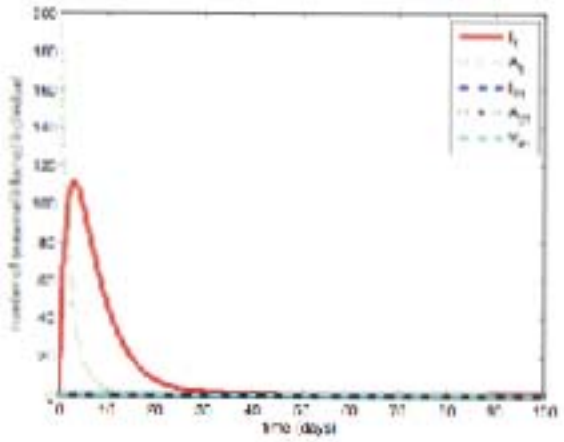
$$\begin{aligned} u_1^* &= min(max(LB_1, -\frac{1}{B_1}(\lambda_1(\beta_1 SJ_1 + \beta_2 SJ_2) + \lambda_2(-\beta_1 SJ_1) + \beta_2 R_1 J_2(\lambda_4 - \lambda_5) \\ &\quad +\lambda_7(-\beta_2 SJ_2) + \beta_1 R_2 J_1(\lambda_9 - \lambda_{10}) + \beta_2 V_s J_2(\lambda_{14} - \lambda_{15}))), UB_1) \\ u_2^* &= min(max(LB_2, \frac{\gamma_2}{B_2}(\lambda_6 I_{12} + I_2(\lambda_8 - \lambda_9) + I_2^*(\lambda_{13} - \lambda_{17}))), UB_2); \end{aligned}$$

5.1 Numerical Results

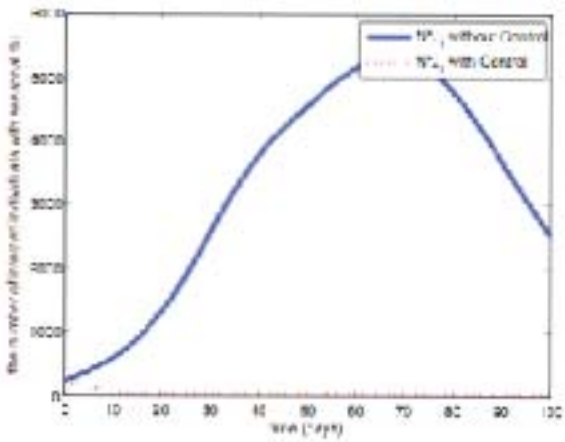
In this section, we analyzed numerically an optimal control strategy on our two-strain influenza model. For the figures presented, we assumed that the number of asymptomatic seasonal individuals is 200 while the number of asymptomatic H1N1 is 100 at time $t = 0$. These figures are divided into two cases: with vaccination and without vaccination. For the simulation, we let the cost of implementing social distancing (B_1=30) be greater than treatment (B_2=10), and we run the simulation for a time span of 100 days.



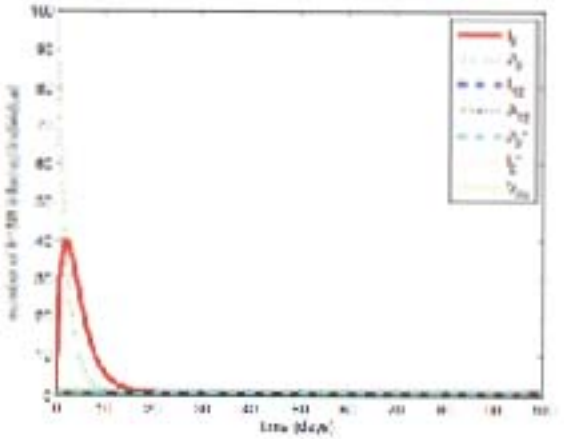
(a) Controls



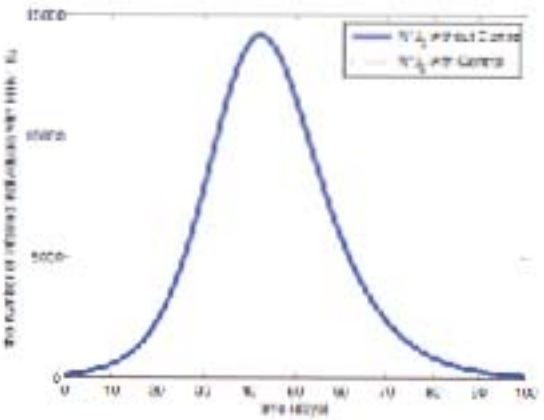
(b) Total Seasonal



(c) Seasonal Infectious classes



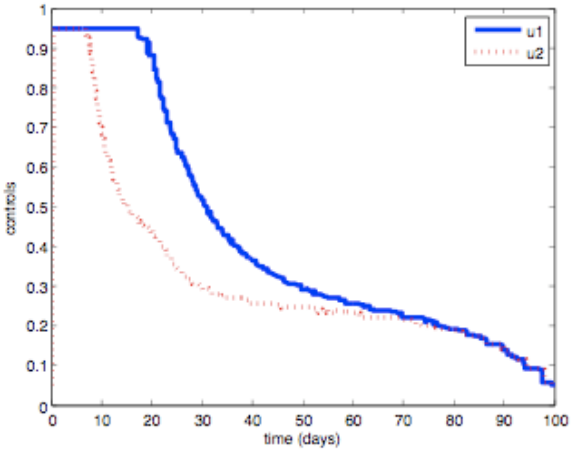
(d) Total H1N1



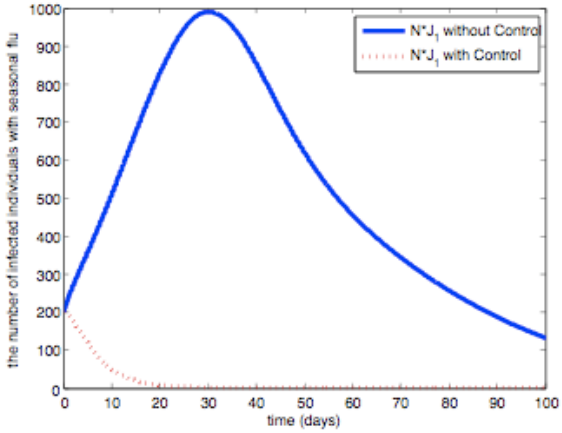
(e) H1N1 Infectious classes

Figure 15: Optimal Control Strategy without Vaccination

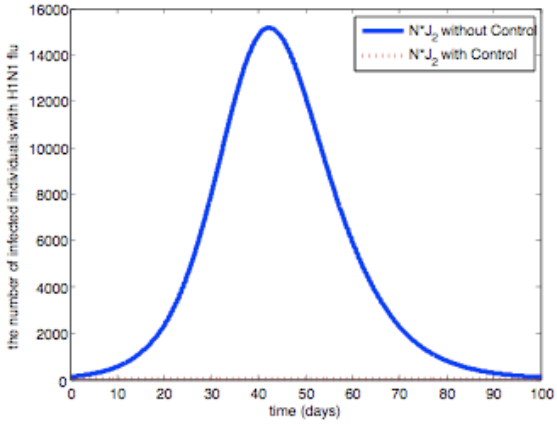
Figure 15 illustrates the optimal control strategy without vaccination. These results showed that in order to reduce the duration and intensity of the outbreak, treatment and social distancing efforts must be kept at maximum efficiency through the peaks of the infectious classes. After that, social distancing remains at 100% for 10 days to ensure that the virus completely dies out while the treatment effort diminishes after the climax. The controls are so effective that they prevent secondary outbreaks from occurring. Since the u_1 control disables the ability of the virus to spread, we can surmise that it is more effective than the u_2 control which can be verified by the plots. If these controls were not applied, the simulation depicts a more significant impact on the severity and duration of the seasonal and H1N1 viruses.



(a) Controls



(b) Seasonal Infectious classes



(c) H1N1 Infectious classes

Figure 16: Optimal Control Strategy with Vaccination

Figure **??** demonstrates the optimal control strategy with vaccination. We noticed that the amount

of effort in the controls decreases at a fairly constant rate, and eventually, they become equivalent before leveling off. The u_2 control effort still remains at optimal level for the same period of time as the model without vaccination, but the u_1 control effort is maintained at a maximum for a shorter time span. Similarly, the presence of control causes a related pattern in the total H1N1 infections regardless of the vaccination factor. However even with the control, the seasonal influenza manages to influence a larger proportion of the infected population. Yet if the controls were not present, the vaccine still plays an important part in minimizing the outbreak of the seasonal flu. When comparing figures **??** and **??**, we observed that in the presence of vaccination, more effort is placed in treatment than when vaccination is absent in our system. This result is consistent with our earlier finding that H1N1 cases increase in the presence of vaccination.

6 Conclusions and Future Work

Simulations were performed on the SAIR Model, the two-strain model and control theory in order to study the effects of the seasonal vaccine and optimal control on an influenza outbreak with seasonal and H1N1 viruses. In the SAIR model, in order to increase the final size in the recovered class, κ needs to be small. This means that if there is a significant number of people that are infected with influenza that don't show symptoms, they will most likely infect more individuals because they are not aware of their infectiousness. Therefore, there will be more people who contract influenza and less end up in the recovered class. The analysis of the SAIR model helped us understand how the viruses interactions occur in the larger, and more complicated two-strain model.

After doing simulations for the two-strain model, the results showed that administering the seasonal vaccine overall reduced the number of infectious individuals in an outbreak. However, it increases the number of H1N1 infections. These results make sense because if people are being vaccinated against seasonal and the transmission rate for H1N1 is higher than for seasonal, the outcome would be more people being infected with H1N1. Also, it was shown that when the initial population of those infected with seasonal is larger than that of H1N1, there is a higher chance of wasting vaccines.

Using optimal control theory determined that implementing treatment and social distancing has a substantial effect on controlling the number of infections during an outbreak. When the social distancing begins, the number of infectious individuals declines rapidly. The controls are so effective in suppressing the spread of infection that they prevented secondary influenza infection cases from forming. Although social distancing is more effective than treatment in controlling the disease, the effort placed in treatment became more important in the presence of vaccination.

In conclusion, social distancing has proven to be the better strategy for controlling seasonal influenza infections, while treatment is more effective when trying to control the H1N1 infections. In the future, we would like to explore ways to improve our model or ask more questions about the movement in the two-strain model. These ideas include incorporating the H1N1 vaccine, hav-

ing individuals move directly from asymptomatic to recovered, including low-risk and high risk susceptibles, and using a different objective function involving only H1N1.

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A Inverse of V

The inverse of V where $A = (\gamma_1 + \mu_1)^{-1}$ and $B = (\gamma_2 + \mu_2)^{-1}$:

$$\begin{bmatrix}
(\alpha_1+\nu)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & {\alpha_1}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\alpha_2+\nu)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & {\alpha_2}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & {\alpha_2}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 & 0 & A & 0 & A & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
\frac{\nu}{(\alpha_1+\nu)(\alpha_1)} & 0 & 0 & 0 & 0 & 0 & 0 & {\alpha_1}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_2}{(\gamma_2+\mu_2)(\alpha_2+\nu)} & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 \\
0 & 0 & \frac{\nu}{\alpha_2(\alpha_2+\nu)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\alpha_2}^{-1} & 0 \\
0 & 0 & \frac{\nu\,(\alpha_2)}{\alpha_2(\gamma_2+\mu_2)(\alpha_2+\nu)} & 0 & B & 0 & 0 & 0 & 0 & 0 & \frac{(\alpha_2)}{\alpha_2(\gamma_2+\mu_2)} & B
\end{bmatrix}$$

B Control Code for SAIR

MATLAB Code for Optimal Control Simulations:

```
%% Euler Method for finding optimal control pair (u1*,u2*)

clc
clear all

tf=100;
MAX = floor(tf/.01)+1;
dt = 0.01;

%% parameters

N=10^5;

b= .2793;
a=.5;
g=.225;
k=0;
gT = g;
m = 0;

%% Upper and lower bounds
LB1 = .05;
UB1 = .95;
LB2 = .05;
UB2 = .95;

%% Weight constants
B1 = 20;
B2 = 10;

%% initialize column vectors
 u1_old = 0.5*ones(MAX,1)';
 u2_old = 0.5*ones(MAX,1)';

u1_old = 0*ones(MAX,1)';
u2_old = 0*ones(MAX,1)';

u1_new = 0*ones(MAX,1)';
u2_new = 0*ones(MAX,1)';

S = zeros(1,MAX)';
A = zeros(1,MAX)';
I = zeros(1,MAX)';
R = zeros(1,MAX)';

l1=0.5*ones(1,MAX)';
l2=0.5*ones(1,MAX)';
l3=0.5*ones(1,MAX)';
l4=0.5*ones(1,MAX)';

l1_new=zeros(1,MAX)';
l2_new=zeros(1,MAX)';
l3_new=zeros(1,MAX)';
l4_new=zeros(1,MAX)';
```

```
%% initial conditions

A(1) = 100;
I(1) = 0;
R(1) = 0;
S(1) = N-A(1)-I(1);

iter = 0;
error = 1;
while error >.001
    iter = iter+1

    %% forward Euler

    for i = 1:MAX-1

        S(i+1)=S(i) + dt*(m*N-b*(1-u1_old(i))*S(i)*(A(i)+I(i))/N-
m*S(i));

        A(i+1)=A(i) + dt*(b*(1-u1_old(i))*S(i)*(A(i)+I(i))/N-
A(i)*(a+k)-m*A(i));

        I(i+1) = I(i) + dt*(a*A(i)-(g +gT*u2_old(i))*I(i)-m*I(i));
    end

    l1(MAX)=0;
    l2(MAX)=0;
    l3(MAX)=0;
    l4(MAX)=0;


       %% backwards Euler

        for i = MAX:-1:2
            l1(i-1)=l1(i) - dt*((l1(i)*(b*(1-u1_old(i))*(A(i)+I(i))/N-
m)+l2(i)*(-b*(1-u1_old(i))*(A(i)+I(i))/N)));

            l2(i-1)=l2(i) - dt*(-(1+l1(i)*(-(b*S(i)*(1-
u1_old(i))/N))+l2(i)*(b*S(i)*(1-u1_old(i))/N-m-(a+k))+l3(i)*(a)));
%+l4(i)*(k)));

            l3(i-1)=l3(i) - dt*(-(1+l1(i)*(-b*S(i)*(1-u1_old(i))/N)+
l2(i)*(b*S(i)*(1-u1_old(i))/N)+l3(i)*(-(g+gT*u2_old(i))-m)));
%+l4(i)*(g+gT*u2_old(i))));
        end


        c=0.9;
    for i=1:MAX

        %% Define u1 and u2
          u1 = min(max(LB1, 1/B1*(b*S(i)*(A(i)+I(i))/N)*(l2(i)-
l1(i))),UB1);
          u2 = min(max(LB2,1/B2*(gT*I(i)*(l3(i)-l4(i)))),UB2);

         %%control updates for u1 using weighted convex combination
```

```
        if (u1 >u1_old(i))
            u1_new(i)=UB1*c^iter+u1_old(i)*(1-c^iter);
        else
            u1_new(i)=LB1*c^iter+u1_old(i)*(1-c^iter);
        end

        %%control updates for u2 using weighted convex combination
        if (u2>u2_old(i))
            u2_new(i)=UB2*c^iter+u2_old(i)*(1-c^iter);
        else
            u2_new(i)=LB2*c^iter+u2_old(i)*(1-c^iter);
        end

    end
        error = max(abs(u1_old-u1_new)+abs(u2_old-u2_new))

        u1_old = u1_new;

        u2_old = u2_new;

end


%% Solve state system without control

x0 = [N-A(1);A(1);0;0];

tspan = 0:0.01:tf;

[WW,z] = ode45(@SAIR2,tspan,x0);

AA = z(:,2);
II = z(:,3);

%% plot
TT = 0:0.01:tf;

l=1;
figure(l)
size(TT)
plot(TT,A'+I','--r','LineWidth',2)
xlabel('time (in days)')
ylabel('Number of Individuals')
legend('With Control')

l = l+1;
figure(l)
plot(tspan,AA+II,'b','LineWidth',2)

xlabel('time (in days)')
ylabel('Number of Individuals')
legend('Without Control')

l=l+1;
figure(l)
plot(TT,u1_old,'b','Linewidth',2)
hold on
```

```
plot(TT,u2_old,'--r','LineWidth',2)

xlabel('time (in days)')
ylabel('Control effort')
legend('u_1','u_2')
```

MATLAB Code for Solving SAIR System Without Control:

```
function [dzdt] =SAIR2(WW,z)

N=10^5;

%% H1N1 parameters
b1= .2793;
a1=.5;
g1=.225;
k1=0;

%% State Equations for SAIR model
dzdt(1)=-b1*z(1)*(z(2)+z(3))/N;

dzdt(2)=b1*z(1)*(z(2)+z(3))/N-z(2)*(a1+k1);

dzdt(3)=a1*z(2)-g1*z(3);

dzdt(4)=g1*z(3)+k1*z(2);

dzdt=dzdt';
```