A New Perspective on Modeling Forest Fires

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Abstract

In this study, we use numerical simulations to heuristically explore the spread of forest fires. Our numerical studies are based on a "bottom-up" framework in which we start with a model with no spatial information on how forest vegetation is distributed (the Mean Field model (MF)). The MF is then replaced by a more detailed model which explores the effects of local, spatial interactions between vegetation and fire (the Pair-Approximation model (PA)). In this detailed study, the MF model serves as our "null" model because of its disconnection from actual biological processes (i.e. the absence of spatial interactions between vegetation and fire and how it affects fire spread). The most developed model in our framework is a Cellular Automata (CA) model. The stochastic and spatially explicit features of the CA model make it ideal for exploring the effects of distance and random behavior on the spread of fire. With the CA model, we gain insight that is directly applicable to actual forest fire management. For each model, we compare and contrast the dynamics of fire spread using a single and two layered (connected) lattice to measure the effect of including differential behavior of fire between the understory and canopy. From each of the models we observe thresholds (when available) for the stability of the fire-free equilibrium (FFE). We also utilize sensitivity analysis to determine the relationships between parameters in the MF and PA model and the basic ignition number, a measure for the average number of new trees that should catch on fire when a single source of fire is introduced into a forest. Results indicate that for all three models, the rate of fire spread (α), the rate at which an occupied burning state returns to a non-burning occupied state (β), and the rate at which a burning occupied state becomes an empty site (γ) determines the stability or instability of a forest fire. In the case of the two-layered lattice versions of the models, we find that fire controls are best focused on the understory level.

1 Introduction

Empirical and theoretical management strategies for minimizing the spread of potentially harmful forest fires have important consequences for the environment and its inhabitants. For example, strategies should be implemented to reduce potentially devastating effects caused by the spread of a forest fire into nearby residential areas. Recently, a number of mathematical models have been developed to predict the spread of forest fires and the environmental consequences that may ensue (Schueller, 2003; Mandel, Beezley, Bennethum, Chakraborty, Coen, Douglas, Hatcher, Kim, and Vodacek, 2007; Michelis and Consolini, 2002; D'Ambrosio, Spataro, and Trunfio, 2006). Many of these models germinated from Rothermel's fire model (1983), which predicted the spread of a forest fire by taking into account biologically relevant parameters such as meteorological trends, fuel characteristics, and forest topography.

In the present study, we use previous theoretical research to develop a multi-layered mathematical modeling approach based within a bottom-up framework to understand the spread of fire and to determine strategies for minimizing its spread and impact.

2 A Multi-Model Approach

Given the complex nature of forest fires, theoretical models that capture their dynamics in a satisfactory manner are sparse. In addition, it is not always clear what the advantages and disadvantages are in using different models (Figure 1). As a result, our study uses several models based within a bottom-up framework to capture the effects of the explicit inclusion of space (or its absence) and stochasticity on the spread of fire through a forest. We first begin with the coarsest model (i.e. the model with the least amount of information about the spatial distribution of vegetation and fire across a forest), a single layer Mean Field (MF) model. This particular model serves as our null model, because of its disconnection from actual biological processes (i.e. the absence of spatial interactions). We add complexity to the MF model by adding a second layer directly above the first. This additional layer not only allows us to study the spread of fire on the crowns of trees, but also the vertical spread of fire between the understory and the canopy of the forest.

Unlike the MF model, the effects of local spatial interactions between vegetation and fire on fire spread through a forest can be explored with the Pair Approximation (PA) model. Analysis of the PA model first begins with the exploration of fire spread on a single layer, then followed by an expansion to include a second layer. However, one of the disadvantages of the PA model is the fact that it loses its ability to track forest fire spread when spatial interactions between vegetation and fire become distant.

In order to explore the effects of space and stochasticity on fire spread, we use a Cellular Automata (CA) model. With the CA model, we gain a sense of the effects of explicit space and stochasticity on fire spread for both a one and two-layered forest. However, since the process uses stochastic simulations, it can become computationally intensive.

In addition to the muti-model approach we also employ a sensitivity analysis. The sensitivity analysis allows us to determine the relationships between parameters in the MF and PA model and the basic ignition number, a measure for the average number of new trees that should catch fire when a single source of fire is introduced into a forest.

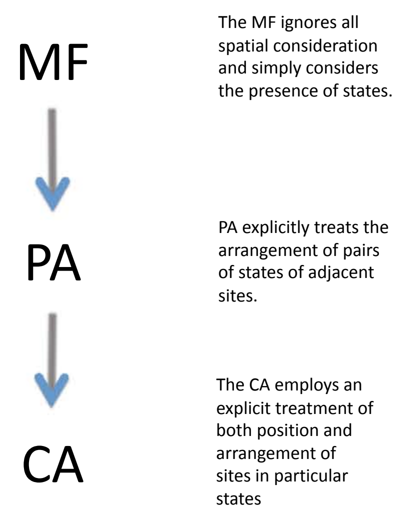

Figure 1: Analysis progresses from a general model to one with more complex considerations. As we move, we have gains and losses in biological relevance.

3 Models

During the derivation of the models, various simplifying assumptions were made. We did not create any distinction between the various types of fuel, the vegetation properties of the forest, or the effects of weather or forest topography. The models utilized in this paper were built upon the differential rates of events during the course of a forest fire (Table **??**. Note: When studying a single layer model, only α_B is taken into consideration since fire can only spread horizontally). During the study all initial conditions were set at random (i.e. the proportion of empty, occupied, and occupied and burning sites). Each site in the lattice may be in one of three states, as denoted by Table **??** and shown in Figure **??**

State	**Definition**
0	Empty site
1	Occupied site
2	Occupied and burning site

Table 1: Description of possible states of sites within the model.

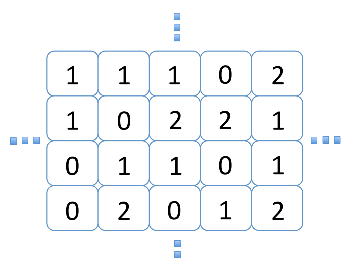

Figure 2: Extract of the lattice

The status of any site on the lattice may transition into another state at a given rate, as described by Figure **??**.

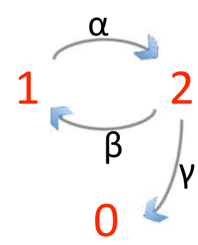

Figure 3: Transitions into other states.

Parameter	Definition
α	Rate at which fire spreads
β	Rate at which a burning state returns to an occupied state
γ	Rate at which a burning state becomes an empty state

With the addition of a second lattice, new rates of spread need to be introduced because the rate at which fire spreads on the canopy of a forest may differ from the rate at which fire spreads on the understory of a forest.

Parameter	Definition
α_B	Rate at which the fire spreads along the bottom layer
α_T	Rate at which the fire spreads along the top layer
α_U	Rate at which the fire ladders up
α_D	Rate at which the fire ladders down

For each parameter we also provide an intervention strategy that we apply appropriately according to the model we are studying.

Parameter	Intervention
Burning vegetation to non-burning vegetation, (β)	Applying a fire retardant
Burning vegetation to an empty site, (γ)	(see future work)
Fire spreads along the understory, (α_B)	Clearing understory fuel
Fire spreads along the canopy, (α_T)	Trimming canopy
Fire ladders up, (α_U)	Trimming dead branches
Fire ladders down, (α_D)	Trimming dead branches

3.1 Differential Equation Models: MF and PA

For our differential equation models we use a notation standard to the Ordinary Pair Approximation literature in order to preserve a continuity between that used in the MF and PA models. Our state variables in each model describe the *probability that a randomly chosen site, on a particular layer, is in a particular state.* This is denoted by $P[i_l]$, where the state considered is i and the layer it occupies is l. We express joint probabilities similarly as

$$P[i_l j_k]$$

and the conditional probability that a site on layer l is in state i given that a site on layer k is in state j is given by

$$Q_{i_l|j_k} := \frac{P[i_l j_k]}{P[j_k]}.$$

3.2 Mean Field Model

Recall that the Mean Field model ignores all spatial arrangement and thus the *probability of any combination of sites being in a particular combination is simply the product of the probabilities of individual sites being in each state* (i.e. $P[i_l j_k] = P[i_l]P[j_k]$). Thus, we may write these equations intuitively as any compartmental model as the sum of all flows in minus the sum of the flows out:

$$\frac{dP[i_l]}{dt} = \sum_{j,k} P[j_k] r_{j_k \to i_l} - P[i_l] \sum_{j,k} r_{i_l \to j_k},$$

where $r_{i_l \to j_k}$ is the rate at which a site on layer l in state i becomes a site on layer k in state j.

$$
\begin{aligned}
\frac{dP[0_B]}{dt} &= \gamma P[2_B]P[0_T] \\
\frac{dP[1_B]}{dt} &= P[2_B](\beta + \gamma P[1_T]) - P[1_B](\alpha_B P[2_B] + \alpha_D P[2_T]) \\
\frac{dP[2_B]}{dt} &= \alpha_B P[1_B]P[2_B] + \alpha_D P[1_B][2_T] - P[2_B]\Big(\beta + \gamma(1 - P[2_T])\Big) \\
\frac{dP[0_T]}{dt} &= \gamma P[2_T] + \gamma P[2_B](1 - P[0_T]) \qquad (1) \\
\frac{dP[1_T]}{dt} &= \beta P[2_T] - P[1_T]\Big((\gamma + \alpha_U)P[2_B] + \alpha_T P[2_T]\Big) \\
\frac{dP[2_T]}{dt} &= P[1_T](\alpha_T P[2_T] + \alpha_U P[2_B]) - P[2_T](\gamma P[2_B] + \gamma + \beta).
\end{aligned}
$$

In system (**??**), we assume that vegetation on the bottom, specifically a tree that reaches the canopy, becomes consumed by fire and that there is some vegetation on the top, which may or may not be burning. We also consider the case where vegetation will fall, resulting in emptying the top and placing its state on the bottom layer.

3.2.1 Determining Conditions for Fire Stability using the Basic Ignition Number (Both layers)

For the MF model, the fire ignition number is denoted by $\mathfrak{F}_0$, and it is a measure of the average number of new trees that would catch on fire when a single source of fire is introduced in a forest. To calculate this quantity, we use Next Generation Operator, which is defined:

$$
FV^{-1} := \begin{pmatrix} F_{BB} & F_{BT} \\ F_{TB} & F_{TT} \end{pmatrix},
$$

where

$$F_{BB} = \frac{\alpha_B}{\gamma+\beta} P[1_B]^*$$
$$F_{BT} = \frac{\alpha_D}{\gamma+\beta} P[1_B]^*$$
$$F_{TB} = \frac{\alpha_U}{\gamma+\beta} P[1_T]^* \tag{2}$$
$$F_{TT} = \frac{\alpha_T}{\gamma+\beta} P[1_T]^*$$

and $P[1_j]^*$ is the amount of occupied sites on layer j before the fire. In (**??**), F_{BB} denotes the fire ignition number that is generated by the burning state along the bottom layer of the lattice, F_{BT} denotes the fire ignition number that is generated by a burning state on the top layer laddering down to the bottom, and F_{TB} denotes the fire ignition number that is generated by a burning state in the bottom layer laddering up to ignite the vegetation on the top layer. Lastly, F_{TT} denotes the fire ignition number that is generated by a burning state on the top layer along the top layer. The basic ignition number is given by the largest eigenvalue of FV^{-1}:

$$\mathfrak{F}_0 = \frac{1}{2}\left(F_{BB} + F_{TT} + \sqrt{(F_{BB} - F_{TT})^2 + 4F_{BT}F_{TB}}\right) \tag{3}$$

$\mathfrak{F}_0$ essentially gives us the stability of the fire-free equilibrium (FFE) for the cases of one and two layers. When we consider spread of fire on only a single layer, we have $\mathfrak{F}_0 = F_{BB} = \frac{\alpha_B}{\gamma+\beta} P[1_B]^*$.

3.2.2 Determining Fire Stability using the Jacobian Matrix (1 layer MF model)

There exists a simpler, albeit weaker condition, for the stability of the FFE. This condition is obtained by analyzing the eigenvalues of the Jacobian at the FFE. Starting with the single layer, the Jacobian matrix is given by:

$$J = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & -P[2_B]\alpha_B & \beta - P[1_B]\alpha_B \\ 0 & P[2_B]\alpha_B & P[1_B]\alpha_B - (\beta+\gamma) \end{pmatrix}. \tag{4}$$

Evaluating the Jacobian matrix at the FFE and excluding the zero eigenvalues yields the following:

$$J|_{FFE} = \begin{pmatrix} 0 & \beta - a\alpha_B \\ 0 & a\alpha_B - (\beta+\gamma) \end{pmatrix}, \tag{5}$$

where $P[0_B]^* = 1-a$, $\lambda_{1,2} = 0$, and $\lambda_3 = a\alpha_B - (\beta+\gamma)$. In order to determine the conditions for stability, we need to set λ_3 less than zero. Thus, the system is stable *iff* $a\alpha_B < \beta + \gamma$.

3.2.3 Stability using the Jacobian (2 layers)

The Jacobian of the system evaluated at the FFE for the double layer version is given by:

$$J|_{FFE} = \begin{pmatrix} 0 & 0 & \gamma(1-b) & 0 & 0 & 0 \\ 0 & 0 & \beta + b\gamma - a\alpha_B & 0 & 0 & -a\alpha_D \\ 0 & 0 & a\alpha_B - (\beta+\gamma) & 0 & 0 & a\alpha_D \\ 0 & 0 & \gamma - (1-a)\gamma & 0 & 0 & \gamma \\ 0 & 0 & -b(\gamma + \alpha_U) & 0 & 0 & \beta - b\alpha_T \\ 0 & 0 & b\alpha_U & 0 & 0 & b\alpha_T - (\gamma+\beta) \end{pmatrix}, \tag{6}$$

where $P[0_B]^* = 1-a$ and $P[0_T]^* = 1-b$.

In order to find the eigenvalues for (6), we need to compute the determinant of the Jacobian. We can reduce the 6x6 matrix to a 2x2 matrix by excluding the zero eigenvalues, which produces the following Jacobian matrix:

$$\hat{J} = \begin{pmatrix} a\alpha_B - (\beta+\gamma) & a\alpha_D \\ b\alpha_U & b\alpha_T - (\gamma+\beta) \end{pmatrix}. \tag{7}$$

The trace of $\hat{J}$ is given by

$$\tau = a\alpha_B + b\alpha_T - 2(\beta+\gamma)$$

and the determinant is given by

$$\Delta = ab(\alpha_B\alpha_T - \alpha_D\alpha_U) - (a\alpha_B + b\alpha_T)(\beta+\gamma) + (\beta+\gamma)^2.$$

If $\tau > 0$, then the FFE is unstable. We may write this condition as

$$a\alpha_B + b\alpha_T > 2(\beta+\gamma) \tag{8}$$

The left-hand side of (8) contains two state variables and two parameters, whereas the right-hand side of the inequality contains only two parameters. This expression makes intuitive sense because if the rate at which occupied spaces catch fire ($P[1_B]^*\alpha_B$) on the bottom and top layer ($P[1_T]^*\alpha_T$) is greater than the rates at which occupied spaces either stop burning (β) or burn down completely (γ), then there is an outbreak of fire.

The relevance of the statements and their corresponding stability, $a\alpha_B + b\alpha_T < 2(\beta+\gamma)$ and $\alpha_B\alpha_T > \alpha_D\alpha_U$, are discussed below. $a\alpha_B+b\alpha_T < 2(\beta+\gamma)$ is just the antithesis of when $\tau > 0$ in the above paragraph. The opposite of $\alpha_B\alpha_T > \alpha_D\alpha_U$, the left-hand side of the inequality, are the rates at which occupied states combust and the right-hand side denotes the rates at which fire scales down a tree (α_D) or up a tree (α_U). In order for stability to hold, the trace has to be negative and the determinant positive. There is the case where $a\alpha_B + b\alpha_T < 2(\beta + \gamma)$, denoted condition one, holds true but the other inequality $\alpha_B\alpha_T > \alpha_D\alpha_U$, referred to as condition two, does not hold true; this would mean the rate at which fire spreads up and down a tree is greater than the rate at which fire spreads along the surface of the forest or the crowns. In this case, the spread of fire is unstable because although the fire would not be an epidemic on either plane, the spread of fire along the trunks of trees can still cause an outbreak of fire. This can be illustrated by a forest fire that initially has a fire on the bottom layer with a given α_B, but once it causes any tree trunk to catch on fire, the fire can quickly reach the crowns and create a crown fire. The more trunks that catch fire, the bigger the possibility of an outbreak occurring. If condition two is met but condition one is not, the system is unstable. If both conditions are not met, this is a saddle point, which is unstable.

3.2.4 Sensitivity Analysis of MF (Single Layer)

Recalling that $\mathfrak{F}_0 = F_{BB} = \frac{\alpha_B P[1_B]^*}{(\gamma+\beta)}$, we are able to do a forward sensitivity analysis, in order to determine which parameters in the model have the greatest effect on forest fire dynamics. In order to study the sensitivity, we must find $\frac{input}{output}\frac{\partial output}{\partial input}$, which can be achieved by calculating $\frac{P}{U}\frac{\partial U}{\partial P}$, where U denotes the function, in our case $\Im_0$, and P denotes the specific parameter we want to study. For example, the sensitivity of α_B is given by

$$\begin{aligned}\frac{P}{U}\frac{\partial U}{\partial P} &= \frac{\alpha_B}{\frac{\alpha_B P[1_B]^*}{(\gamma+\beta)}}\frac{P[1_B]^*}{(\gamma+\beta)} \\ &= 1\end{aligned}$$

Since the sensitivity index is positive, we can say it is directly proportional (i.e. as we increase the value of α_B, the spread of the fire increases). For γ, the sensitivity index is given by $\frac{-\gamma}{\gamma+\beta}$. This formula illustrates that there is an inverse relationship between γ and the spread of fire. For β, the sensitivity index is given by $\frac{-\beta}{\gamma+\beta}$. Again, we see that there is an inverse relationship, such that

$$\begin{aligned} S_{\alpha_B} &= 1 \\ S_{\gamma} &= \frac{-\gamma}{\gamma+\beta} \\ S_{\beta} &= \frac{-\beta}{\gamma+\beta}. \end{aligned}$$

These results show that the affect of a percent change in α_B would be nullified by a percent change in both γ and β.

3.2.5 Sensitivity Analysis of MF (Double Layer)

For the derivation of the sensitivity indices of the MF Double Layer, we are able to utilize the chain rule to simplify our derivation of the sensitivity indices for $\mathfrak{F}_0$. As seen below, the sensitivity index for parameter P is given by

$$\frac{P}{U}\frac{\partial U}{\partial P} = \frac{P}{U}\left(\frac{\partial \Im_0}{\partial F_{BB}}\frac{\partial F_{BB}}{\partial P} + \frac{\partial \Im_0}{\partial F_{BT}}\frac{\partial F_{BT}}{\partial P} + \frac{\partial \Im_0}{\partial F_{TB}}\frac{\partial F_{TB}}{\partial P} + \frac{\partial \Im_0}{\partial F_{TT}}\frac{\partial F_{TT}}{\partial P}\right)$$

Further details on the sign of each sensitivity may be found in Appendix B.

3.2.6 Single Layer Forest

The MF model gave us a mathematical means for determining a FFE. A stable FFE is achieved by keeping $\mathfrak{F}_0 = \frac{P[1_B]\alpha_B}{(\beta+\gamma)} < 1$, and unstable by keeping $\mathfrak{F}_0 = \frac{P[1_B]\alpha_B}{(\beta+\gamma)} > 1$. As described by the inequality, in order to achieve a stable FFE it is necessary to keep the product of the available rate of the spread of fire and the proportion of non-burning vegetation less than the rates in which fires return to non-burning states.

We discuss the behavior of the fire for the MF model by studying different cases for fire stability and instability, and interpret the results generated from each case separately.

a) Stability cases

For the MF model, a particular property that should be observed is the behavior of the proportion of burning states vs. time. The function demonstrates characteristics of exponential decay (Figure 4).

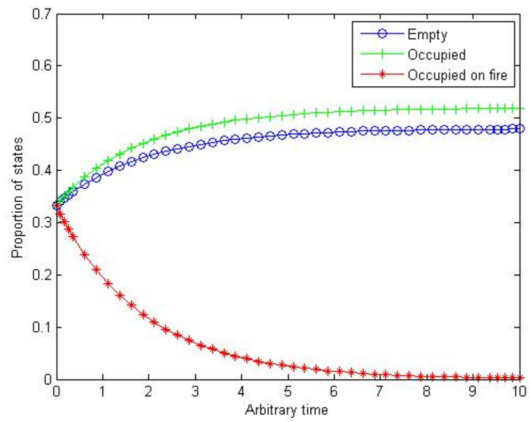

Figure 4: First case of stability, with parameters $\alpha_B = 0.25$, $\beta = 0.4$, $\gamma = 0.23$

If the parameter γ is greater than β, the proportion of empty spaces is larger than that of occupied sites (Figure 5).

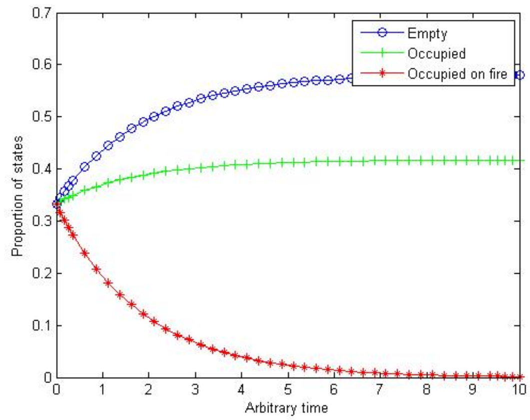

Figure 5: Second case of stability, with parameters $\alpha_B = 0.25$, $\beta = 0.23$, $\gamma = 0.4$

b) Instability case

In the case of instability, we can see that fire demonstrates different behavior; the function reaches a maximum before decreasing to zero (Figure **??**).

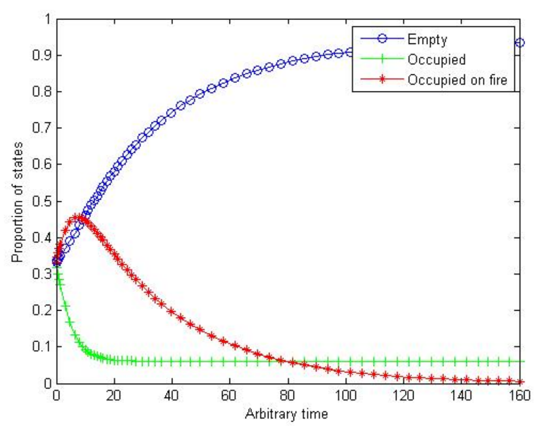

Figure 6: Case of instability, with parameters $\alpha_B = 0.5$, $\beta = 0.03$, $\gamma = 0.03$

3.2.7 MF (Double Layer)

As noted earlier, we can study the effects of different values of the parameter α on the spread of fire through the forest. These include: α_B, the rate at which fire spreads along the understory of the forest; α_T, the rate at which the fire spreads along the crown of the forest; α_D, the rate at which fire spreads from the top to the bottom of the trees; and α_U, the rate at which fire spreads from the bottom to the top of the trees. If

$$a\alpha_B + b\alpha_T < 2(\beta + \gamma) \text{ and } \alpha_B\alpha_T > \alpha_D\alpha_U,$$

then there is a stable FFE.

The condition states that a stable FFE can be reached when the sum of the rates at which fire spreads along the bottom and top of the forest must be less than two times the sum at which the fire moves from a burning to non-burning state. This condition suggests that if the rate of the vertical spread of fire is greater than the rate of horizontal spread, an unstable FFE will be the result (Figures **??** and **??**).

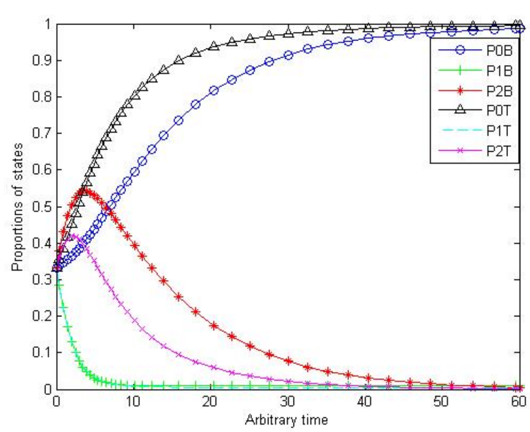

Figure 7: First case of instability, $\alpha_B = 0.95$, $\alpha_T = 0.7$, $\alpha_D = 0.3$, $\alpha_U = 0.4$, $\beta = 0.009$, $\gamma = 0.09$

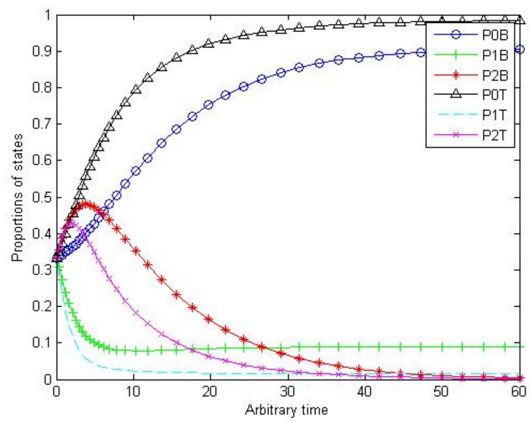

Figure 8: Second case of instability, $\alpha_B = 0.3$, $\alpha_T = 0.45$, $\alpha_D = 0.7$, $\alpha_U = 0.95$, $\beta = 0.05$, $\gamma = 0.09$

3.3 PA model

As noted before, we use the PA model to capture the dynamics of a forest fire by looking at the interactions between two adjacent states (i.e. local interactions). We do this by first considering every possible combination of pairs of sites as illustrated below:

Derivation of the PA equations (Both layers)
We start by considering all the possible combinations of states:

$$\begin{array}{cccccc}
0_B0_B & 0_B1_B & 0_B2_B & 0_T0_T & 0_T1_T & 0_T2_T \\
1_B0_B & 1_B1_B & 1_B2_B & 1_T0_T & 1_T1_T & 1_T2_T \\
2_B0_B & 2_B1_B & 2_B2_B & 2_T0_T & 2_T1_T & 2_T2_T \\
0_T0_B & 0_T1_B & 0_T2_B & 0_B0_T & 0_B1_T & 0_B2_T \\
1_T0_B & 1_T1_B & 1_T2_B & 1_B0_T & 1_B1_T & 1_B2_T \\
2_T0_B & 2_T1_B & 2_T2_B & 2_B0_T & 2_B1_T & 2_B2_T
\end{array}$$

Since the ODEs of PA model are all built upon probability, we can make the argument that pairs of states such as $[1_B0_B]$ and $[0_B1_B]$ are symmetric. It is because of this feature that we can use the same ODE to represent the same pair of sites.

For the bottom layer:

$$\begin{array}{ll} 1_B 0_B & \\ 2_B 0_B & 2_B 1_B \end{array}$$

For the middle layer:

$$\begin{array}{lll} 0_T 0_B & 0_T 1_B & 0_T 2_B \\ 1_T 0_B & 1_T 1_B & 1_T 2_B \\ 2_T 0_B & 2_T 1_B & 2_T 2_B \end{array}$$

For the top layer:

$$\begin{array}{ll} 1_T 0_T & \\ 2_T 0_T & 2_T 1_T \end{array}$$

When dealing specifically with the middle equations, two more terms can be eliminated, $0_B 1_T$ and $0_B 2_T$. We are able to eliminate these because of the lack of biological information that can be derived from these terms. For example, $0_B 1_T$ states that in a given space on the lattice, the top layer is occupied but there is no occupation in the cell directly beneath the occupied site in the top layer. Clearly, this is not of biological relevance.

By taking into account the law of total probability, we can rewrite $P[2_B 2_B]$ in terms of the other fire states on the bottom layer of the forest (i.e. $P[2_B 2_B] = 1 - P[0_B 0_B] - 2P[0_B 1_B] - 2P[0_B 2_B] - P[1_B 1_B] - 2P[1_B 2_B]$). The same holds true for $P[2_T 2_T]$ and $P[2_B 2_T]$. However, for $P[2_B 2_T]$, there are six other states. $P[2_B 2_T]$ is rewritten as $P[2_B 2_T] = 1 - P[0_B 0_T] - P[1_B 0_T] - P[1_B 1_T] - P[1_B 2_T] - P[2_B 0_T] - P[2_B 1_T]$. After excluding the equations due to symmetry and rewriting $P[2_B 2_B]$, $P[2_T 2_T]$, $P[2_B 2_T]$ in terms of each layer's related states, the following states remain.
For the bottom layer:

$$\begin{array}{lll} 0_B 0_B & 0_B 1_B & 0_B 2_B \\ & 1_B 1_B & 1_B 2_B \end{array}$$

For the middle layer:

$$\begin{array}{lll} 0_B 0_T & & \\ 1_B 0_T & 1_B 1_T & 1_B 2_T \\ 2_B 0_T & 2_B 1_T & \end{array}$$

For the top layer:

$$\begin{array}{lll} 0_T 0_T & 0_T 1_T & 0_T 2_T \\ & 1_T 1_T & 1_T 2_T \end{array}$$

The derivation of the ODEs is intuitive. For example, if we were to pick pair of states such as 1_B and 1_B, which denote two non-burning, adjacent occupied spaces, then by adding the possible inflow and subtracting the possible outflow, as in the MF systems, we derive the respective ODE. In the case of $[1_B 1_B]$, the possible inflows are:

$$[2_B 1_B] \text{ and } [1_B 2_B],$$

and the possible outflows are:

$$[1_B 2_B] \text{ and } [2_B 1_B].$$

As mentioned previously, the rate of change of the probability of selecting a pair of sites in the arrangement $[1_B 1_B]$ is

$$\frac{dP[1_B 1_B]}{dt} = \sum inflow - \sum outflow$$

which is described by the equation:

$$\begin{aligned} \frac{dP[1_B 1_B]}{dt} &= 2P[1_B 2_B](\beta + \gamma Q_{1_T|2_B}) \\ &\quad -2P[1_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \alpha_D Q_{2_T|1_B}\right), \end{aligned} \tag{9}$$

As noted earlier, the PA model includes the coupling of pairs consisting of a site from the bottom and the site directly on top. Given a state in $[1_B 2_T]$, the possible inflows are given by:

$$[1_B 1_T] \text{ and } [2_B 2_T],$$

and the possible outflow is given by:

$$[2_B 2_T],\ [1_B 0_T],\ \text{and } [1_B 1_T].$$

This leads directly to a derivation of the ODE for the pair $[1_B 2_T]$:

$$\frac{dP[1_B 2_T]}{dt} = P[1_B 1_T]\left(\alpha_T Q_{2T|1T} + P[2_B 2_T]\beta\right) - P[1_B 2_T]\left(\gamma + \alpha_B Q_{2B|1B} + \alpha_D + \beta\right)$$

In this particular equation, we also considered the possibility of the neighbor to $[1_B 2_T]$ being on fire on the bottom, $Q_{2_B|1_B}$. As a result, we may see a state change from $[1_B 2_B] \to [2_B 2_T]$.

Lastly, given a pair of sites in a top layer such as $[0_T 0_T]$, we can derive a top equation:

$$[0_T 2_T],\ [2_T 0_T],\ [0_T 1_T],\ [1_T, 0_T]$$

Using this process, we can now list all the equations for all parts of the two-layer PA model.

Bottom equations:

$$\begin{aligned}
\frac{dP[0_B 0_B]}{dt} &= 2\gamma P[2_B 0_B] Q_{0_T|2_B} \\
\frac{dP[0_B 1_B]}{dt} &= P[0_B 2_B](\beta + \gamma Q_{1_T|2_B}) + \gamma P[1_B 2_B] Q_{0_T|2_B} \\
&- P[0_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \alpha_D Q_{2_T|1_B}\right) \\
\frac{dP[0_B 2_B]}{dt} &= P[0_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \alpha_D Q_{2_T|1_B}\right) + \gamma P[2_B 2_B] Q_{0_T|2_B} \\
&- P[0_B 2_B]\left(\gamma Q_{1_T|2_B} + \beta + \gamma Q_{0_T|2_B}\right) \\
\frac{dP[1_B 1_B]}{dt} &= 2P[2_B 1_B](\beta + \gamma Q_{1_T|2_B}) - 2P[1_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \alpha_D Q_{2_T|1_B}\right) \\
\frac{dP[1_B 2_B]}{dt} &= P[2_B 2_B](\beta + \gamma Q_{1_T|2_B}) + P[1_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \alpha_D Q_{2_T|1_B}\right) \\
&- P[1_B 2_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \frac{1}{4}\alpha_B + \alpha_D Q_{2_T|1_B} + \gamma Q_{0_T|2_B}\right. \\
&+ \left.\beta + \gamma Q_{1_T|2_B}\right)
\end{aligned} \tag{10}$$

Middle equations:

$$
\begin{aligned}
\frac{dP[0_B0_T]}{dt} &= P[2_B0_T]\gamma \\
\frac{dP[1_B0_T]}{dt} &= P[2_B0_T]\beta + P[1_B2_T]\gamma + P[2_B1_T]\gamma - P[1_B0_T](\alpha_B Q_{2_B|1_B}) \\
\frac{dP[2_B0_T]}{dt} &= P[1_B0_T](\alpha_B Q_{2_B|1_B}) + 2P[2_B2_T]\gamma - P[2_B0_T](\beta+\gamma) \\
\frac{dP[1_B1_T]}{dt} &= P[2_B1_T]\beta + P[1_B2_T]\beta - P[1_B1_T](\alpha_B Q_{2_B|1_B} + \alpha_T Q_{2_T|1_T}) \qquad (11) \\
\frac{dP[2_B1_T]}{dt} &= P[2_B2_T]\beta + P[1_B1_T](\alpha_B Q_{2_B|1_B}) - P[2_B1_T](\alpha_T Q_{2_T|1_T} + \alpha_U + \beta + \gamma) \\
\frac{dP[1_B2_T]}{dt} &= P[1_B1_T](\alpha_T Q_{2_T|1_T}) + P[2_B2_T]\beta - P[1_B2_T](\gamma + \alpha_B Q_{2_B|1_B} + \alpha_D + \beta)
\end{aligned}
$$

Top equations:

$$
\begin{aligned}
\frac{dP[0_T0_T]}{dt} &= 2P[0_T2_T](\gamma + \gamma Q_{2_B|2_T}) + 2P[0_T1_T](\gamma Q_{2_B|1_T}) \\
\frac{dP[0_T1_T]}{dt} &= P[2_T1_T](\gamma + \gamma Q_{2_B|2_T}) + P[1_T1_T]\gamma Q_{2_B|1_T} + P[0_T2_T]\beta \\
&- P[0_T1_T]\left(\frac{3}{4}\alpha_T Q_{2_T|1_T} + \alpha_U Q_{2_B|1_T} + \gamma Q_{2_B|1_T}\right) \\
\frac{dP[0_T2_T]}{dt} &= P[0_T1_T]\left(\alpha_U Q_{2_B|1_T} + \frac{3}{4}\alpha_T Q_{2_T|1_T}\right) + P[1_T2_T](\gamma Q_{2_B|1_T}) + P[2_T2_T](\gamma + \gamma Q_{2_B|2_T}) \\
&- P[0_T2_T](\gamma + \beta + \gamma Q_{2_B|2_T}) \\
\frac{dP[1_T1_T]}{dt} &= 2\beta P[2_T1_T] - 2P[1_T1_T]\left(\frac{3}{4}\alpha_T Q_{2_T|1_T} + \alpha_U Q_{2_B|1_T} + \gamma Q_{2_B|1_T}\right) \qquad (12) \\
\frac{dP[1_T2_T]}{dt} &= \beta P[2_T2_T] + P[1_T1_T]\left(\frac{3}{4}\alpha_T Q_{2_T|1_T} + \alpha_U Q_{2_B|1_T}\right) \\
&- P[1_T2_T]\left(\gamma Q_{2_B|1_T} + \gamma + \gamma Q_{2_B|2_T} + \beta + \frac{3}{4}\alpha_T Q_{2_T|1_T} + \alpha_U Q_{2_B|1_T} + \frac{\alpha_T}{4}\right)
\end{aligned}
$$

Derivation of the PA equations (1 layer)

In order to derive the single layer equations, one needs only consider the events occurring on the bottom layer. The single layer equations can be written as:

$$
\begin{aligned}
\frac{dP[0_B 0_B]}{dt} &= 2\gamma P[2_B 0_B] \\
\frac{dP[0_B 1_B]}{dt} &= P[0_B 2_B]\beta + \gamma P[1_B 2_B] \\
&- P[0_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B}\right) \\
\frac{dP[0_B 2_B]}{dt} &= P[0_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B}\right) + \gamma P[2_B 2_B] \\
&- P[0_B 2_B]\left(\beta + \gamma\right) \\
\frac{dP[1_B 1_B]}{dt} &= 2P[2_B 1_B]\beta - 2P[1_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B}\right) \\
\frac{dP[1_B 2_B]}{dt} &= P[2_B 2_B]\beta + P[1_B 1_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B}\right) \\
&- P[1_B 2_B]\left(\frac{3}{4}\alpha_B Q_{2_B|1_B} + \frac{1}{4}\alpha_B + \gamma + \beta\right).
\end{aligned} \tag{13}
$$

$P[2_B 2_B]$ is not shown because it can be written in terms of the other five states.

Stability using the Jacobian (1 layer)

We start the discussion of the stability of the FFE with the single layer system. In order to find the FFE, it is necessary to set any pairs that contain a 2 state equal to zero. Noting that all states on the single layer must sum to 1 we get:

$$
\begin{aligned}
P[0_B 0_B] &= P^*[0_B 0_B] \\
P[0_B 1_B] &= P^*[0_B 1_B] \\
P[0_B 2_B] &= P[1_B 2_B] = P[2_B 2_B] = 0 \\
P[1_B 1_B] &= P^*[1_B 1_B] = 1 - P^*[0_B 0_B] - 2P^*[0_B 1_B]
\end{aligned}
$$

Using the Jacobian and determining the eigenvalues we are able to compute the stability of the system. The Jacobian matrix evaluated at the FFE is

$$
J = \begin{bmatrix}
0 & 0 & 2\gamma & 0 & 0 & 0 \\
0 & 0 & \beta & 0 & \gamma - \frac{3\alpha_B P^*[0_B 1_B]}{4P^*[1_B]} & 0 \\
0 & 0 & -(\beta+\gamma) & 0 & \frac{3\alpha_B P^*[0_B 1_B]}{4P^*[1_B]} & \gamma \\
0 & 0 & 0 & 0 & 2\beta - \frac{3\alpha_B P^*[1_B 1_B]}{2P^*[1_B]} & 0 \\
0 & 0 & 0 & 0 & -\frac{\alpha_B}{4} + \frac{3\gamma P^*[1_B 1_B]}{4P^*[1_B]} - (\beta+\gamma) & \beta \\
0 & 0 & 0 & 0 & \frac{\alpha_B}{2} & -2(\beta+\gamma)
\end{bmatrix}.
$$

We can work with a 2x2 matrix by excluding the zero eigenvalues and the negative eigenvalue, $-(\beta+\gamma)$, which produces the following Jacobian matrix:

$$
A = \begin{pmatrix}
-\frac{\alpha_B}{4} + \frac{3\gamma P^*[1_B 1_B]}{4P^*[1_B]} - (\beta+\gamma) & \beta \\
\frac{\alpha_B}{2} & -2(\beta+\gamma)
\end{pmatrix}.
$$

In order to determine stability, we want to know when $b = Tr(A) < 0$ and when $c = Det(A) > 0$. Solving these inequalities yields the following conditions:

$$
b < 0 \quad \Leftrightarrow \quad \frac{P^*[1_B 1_B]}{P^*[1_B]} < \frac{\alpha_B + 12(\beta+\gamma)}{3\alpha_B} \tag{14}
$$

$$
c > 0 \quad \Leftrightarrow \quad \frac{P^*[1_B 2_B]}{P^*[1_B]} < \frac{\alpha_B \gamma + 4(\beta+\gamma)^2}{3\alpha_B(\beta+\gamma)} \tag{15}
$$

Following the work shown in Appendix A, we find the two eigenvalues are real and negative, and since the other eigenvalues are negative, we can conclude the FFE is stable.

Stability using the Basic Ignition Number (1 Layer)
For the stability of the PA single layer we also used the next generation operator, including $\frac{dP[2_B 2_B]}{dt}$.

$$\begin{aligned} P[0_B 0_B] &= P[0_B 0_B]^* \\ P[0_B 1_B] &= P[0_B 1_B]^* \\ P[0_B 2_B] &= P[1_B 2_B] = P[2_B 2_B] = 0 \\ P[1_B 1_B] &= P[1_B 1_B]^* = 1 - P[0_B 0_B]^* - 2P[0_B 1_B]^* \end{aligned}$$

Using the next generation operator we found that the $\mathfrak{F}_\mathfrak{o}$ is given by

$$\mathfrak{F}_\mathfrak{o} = \frac{\alpha_B}{P[1_B]^*} \frac{3\gamma P[1_B 1_B]^* + 4\beta P[1_B]^*}{(\beta+\gamma)(4(\beta+\gamma)+\gamma_B)}.$$

Interpretation of the $\mathfrak{F}_\mathfrak{o}$
We want to know when $\mathfrak{F}_\mathfrak{o} < 1$, i.e.

$$\frac{\alpha_B}{P[1_B]^*} \frac{3\gamma P[1_B 1_B]^* + 4\beta P[1_B]^*}{(\beta+\gamma)(4(\beta+\gamma)+\gamma_B)} < 1 \tag{16}$$

The equation (16) can be written as

$$\alpha_B(3\gamma P[1_B 1_B]^* + (3\beta - \gamma)P[1_B]^*) < 4P[1_B]^*(\beta+\gamma)^2. \tag{17}$$

If $3\gamma P[1_B 1_B]^* + (3\beta - \gamma)P[1_B]^* > 0$ and writing this condition as:

$$\frac{\gamma - 3\beta}{3\gamma} < \frac{P[1_B 1_B]^*}{P[1_B]^*}, \tag{18}$$

then, from (**??**), (18) we have

$$\alpha_B < \frac{4P[1_B]^*(\beta+\gamma)^2}{3\gamma P[1_B 1_B]^* + (3\beta-\gamma)P[1_B]^*}. \tag{19}$$

So, if (**??**) and (**??**) are true, then $\mathfrak{F}_\mathfrak{o} < 1$ and the FFE will be stable (i.e the fire will die out quickly). It is interesting to note that the condition (**??**) depends on the initial proportion of occupied spaces and the values of γ and

β. Based on this information, (**??**) we can choose the value for α_B that gives rise to the FFE.

Note that if $3\gamma P[1_B 1_B]^* + (3\beta - \gamma)P[1_B]^* < 0$, we obtain the following result:

$$\frac{P[1_B 1_B]^*}{P[1_B]^*} < \frac{\gamma - 3\beta}{3\gamma} \Rightarrow \alpha_B > \frac{4P[1_B]^*(\beta+\gamma)^2}{3\gamma P[1_B 1_B]^* + (3\beta - \gamma)P[1_B]^*}.$$

As we can observe, there is a relationship between the rates of states and the initial proportion of occupied spaces.

3.4 Cellular Automata

In order to capture the dynamics of a burning forest, we created a computer simulation consisting of a 100x100 lattice(s). This enables us to analyze the spread of a forest fire in a spatially explicit environment. The composition of the forest fire on the lattice consists of three different states: empty states, occupied states, or occupied and burning states. In order to observe a change in status, the sites were assigned a color: black represents empty site; green represents occupied site; and red represents an occupied site that is on fire. The simulation begins by creating a forest where each cell has equal chance of being in one of the three states. The CA model then proceeds to recreate the spread of fire from the states that are occupied and burning (red) for the duration of the outbreak. The spread of fire is also done in an unbiased manner. If an occupied site is on fire, the fire can spread from the site in a modified von Neumann neighborhood. In the context of fire spreading, the von Neumann neighborhood allows the fire to move in one of four directions: North, South, East, and West. However, in the case of two layers CA model, the fire is allowed to move in the vertical direction. The vertical movement of fire is only possible for trees that are on fire. The spread of the fire is determined by the von Neumann process (Figure **??**).

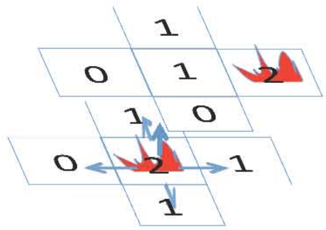

Figure 9: Fire from vegetation can spread North, South, East, West,Up, and/or Down (when applicable)

3.4.1 The Study

Due to the stochastic nature of the CA model, there is no analytic method to determine $\Im_0$. As a result, we approached the problem by choosing random initial conditions, and using various parameter values. Some of the parameters we choose are inspired by the $\Im_0$ calculations from the MF and PA models. By taking permutations of these parameters, we were able to show how combinations of parameter values affect the dynamics of the system. In order to achieve the general solution, we ran the simulation 1000 times for each set of parameter values. In order to differentiate between a stable fire-free equilibrium and an unstable fire-free equilibrium, we state some definitions:

Definition 1. *A fire free equilibrium of the CA is called* ***Stable*** *if and if only the final number of non-burning occupied sites is greater than the initial number of non-burning occupied sites.*

Definition 2. *A fire free equilibrium of the CA is called* ***Unstable*** *if and only if the final number of non-burning occupied sites is less than the initial number of non-burning occupied sites.*

3.4.2 Single Layer

Based on the definition of an unstable FFE for the CA model we were able to determine which parameters allow for such an outcome.

α_B	β	γ	stability
.163	.05	.005	stable
.163	.005	.05	unstable
.165	.05	.005	stable
.165	.005	.05	unstable
.5	.5	.5	stable
.5	.7	.2	stable
.5	.3	.3	stable

There exist several combinations of parameter values that allow for the maintenance of a stable outbreak of fire. Possible conditions of parameters that allow for a stable forest fire include, keeping α as minuscule as possible and keeping β much greater than γ. Although this process is never absolute in outcome, due to stochasticity of events, we can study the general behavior by considering the simulation, repeated 1000 times with the same conditions, as shown in Figure **??**.

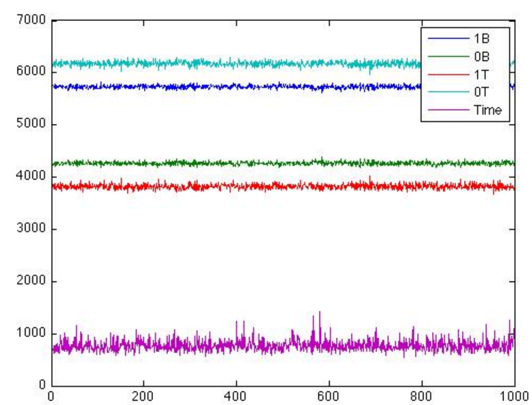

Figure 10: Stable 1, x-axis denotes the *nth* simulation, y-axis denotes the proportion of sites in a given state

The particular example shown above can refer to a situation where the spread of fire is contained within the ignited trees, and a constant fire retardant is being applied to the forest in order to save some vegetation, with the exception of vegetation that is burned to the ground. This particular combination of parameters seems ideal because it shortens the duration of the fire outbreak. Through experimentation and permutation of parameters, we were able to anticipate which parameters are key in keeping the concentration of vegetation high regardless of the size of the fire. Of particular importance is the parameter responsible for the loss of occupied space, γ. Since maintaining a high α_B rate does not result in tree loss, we can minimize the impact of fire on any forest by keeping β much higher than γ.

3.5 Double layer simulations

Like the single layer simulations, values were assigned to our parameters based on results from the MF and PA models. It should be noted that it is possible to have a partially stable system, in which one layer of the lattice ends with more non-burning occupied sites than it initially had.

α_B	α_T	α_U	α_D	β	γ	stability
.163	.163	.163	.163	.005	.05	partially stable
.165	.165	.165	.165	.05	.005	unstable
.082	.082	.082	.082	.05	.005	Stable
.082	.082	.082	.082	.005	.05	unstable
.083	.083	.083	.083	.05	.005	stable
.7	.1	.1	.1	1	.1	stable
.1	.7	.1	.1	1	.1	stable
.1	.1	.7	.1	1	.1	stable
.1	.1	.1	.7	1	.1	stable
.7	.1	.1	.1	.1	1	stable understory
.1	.7	.1	.1	.1	1	stable understory
.1	.1	.7	.1	.1	1	stable understory
.1	.1	.1	.7	.1	1	stable understory
1	1	1	1	1	1	stable understory
.1	.123	.055	.044	.05	.005	Stable
.1	.123	.055	.044	.005	.05	Unstable
.1	.1	.75	.25	.05	.005	stable understory
.1	.1	.25	.75	.05	.005	stable understory
.1	.1	.75	.25	.005	.05	unstable
.1	.1	.25	.75	.005	.05	unstable
.75	.25	.1	.1	.05	.005	unstable
.25	.75	.1	.1	.05	.005	stable understory
.75	.25	.1	.1	.005	.05	unstable
.25	.75	.1	.1	.005	.05	stable understory

From our analysis, we have determined that given a particular double layer lattice we can have instability in one lattice and stability in another. Another phenomenon that was observed from analysis of the CA model was the general result; if we start with a high rate of fire spread along the bottom lattice, this will most likely result in an unstable FFE. This phenomenon suggests that during intervention the best method for preventing fire spread is to implement prevention strategies at the bottom layer of the forest. If there is high fire spread along the understory, this will result in major tree loss, compared to the scenario where fire spread is greater along the canopy. Figure **??** is an example of a stable, FFE in the CA model, while Figure **??** is an example of an unstable FFE.

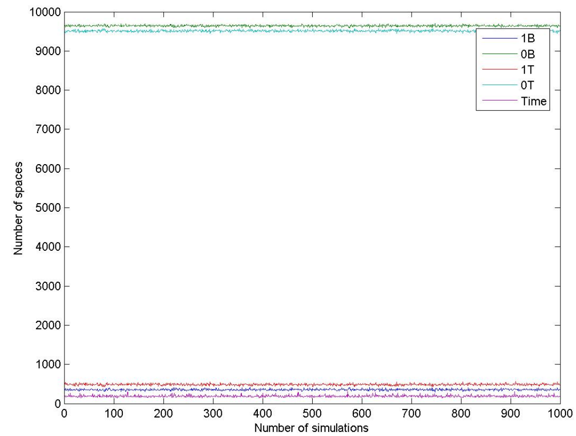

Figure 11: First case of stability in CA, with parameters $\alpha_B = 0.1$, $\alpha_T = 0.123$, $\alpha_D = 0.044$, $\alpha_U = 0.055$, $\beta = 0.05$, $\gamma = 0.005$. Both states 0_B and 0_T are greater than the empty sites in their lattice.

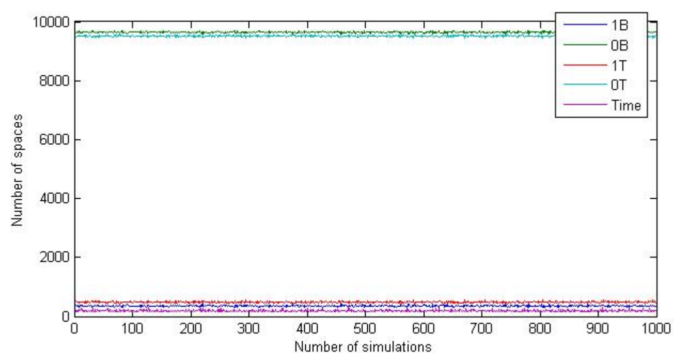

Figure 12: Unstable Fire, with parameters $\alpha_B = 0.25$, $\alpha_T = 0.75$, $\alpha_D = 0.1$, $\alpha_U = 0.1$, $\beta = 0.005$, $\gamma = 0.05$. Both states $0B$ and $0T$ are greater than the occupied sites in their lattice.

In the Figure **??**, we can see the effects of having different values for α_i (where i denotes B, T, U, D).

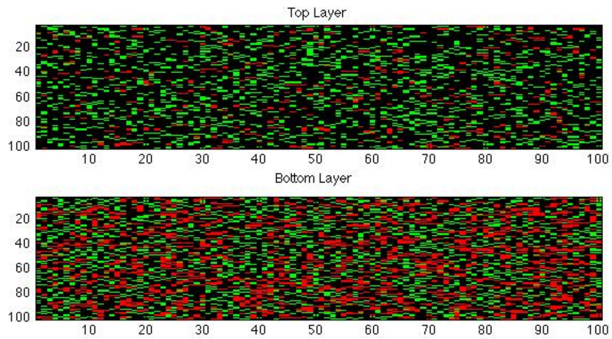

Figure 13: Simulation of a fire

3.6 Discussion and Summary

During the course of this study we used a multi-layered modeling approach based within a bottom-up framework to explore the dynamics of fire spread through a forest. As indicated, each particular model had its advantages and disadvantages.

The MF equations gave us an analytic result for the stability of forest fires. The $\Im_0$, derived from the two layer MF model, demonstrated that having a stable fire in each layer of the forest did not imply the presence of a stable FFE. This result demonstrated that fire spread between two lattices can easily disrupt a stable fire. In addition, this demonstrated that an intervention strategy might minimize fire spread between the canopy and the understory. This could be achieved by trimming tree trunks so that fire does not ladder up and down the tree as easily.

The PA's $\Im_0$ demonstrated that when spatial interaction between adjacent pairs of sites is taken into account, adjacent pairs of occupied, non-burning vegetation contribute to the PA's $\Im_0$. From this result, one could conclude that preventive action should occur in densely occupied locations. The CA model allowed us to study a spatially explicit stochastic model. This allowed us to gain insight into how different combinations of parameters could create longer or shorter durations of fires. Through analysis of the CA model we observed that a large α paired with low values of β and γ resulted in longer outbreaks of fire. An intervention strategy for this particular situation would entail increasing the amount of flame retardant that is applied to the fire. If enough retardant is applied then β should be greater than α, which would result in recovery of more trees, and eventual extinction of the fire. Thus, the larger the value of β, the smaller the time needed to extinguish the fire.

One advantage of having a double layer CA model is that it makes the study of fire spread between two lattices possible. We observed that if $\alpha_B >> \alpha_T$, there will be a larger loss of vegetation. Alternatively, if $\alpha_T >> \alpha_B$, this will result in a less severe loss. These observations emphasize that when taking preventive actions, the best place to start is the understory of the forest. Differences between the one layer and two layer models demonstrate that adding another lattice to create distinction between the understory and the canopy provide better strategies for achieving a stable FFE; a result that cannot be provided by simply studying a single layer model.

4 Future Work

The models presented in this paper are very selective. The prominent assumptions made in this study were the exclusion of weather conditions, fuel properties, and topographical features of the forest. Weather conditions can involve the presence of wind, which can vary in velocity, and/or rain, which can also vary in quantity and velocity. The fuel properties involve the type of vegetation in the forest, which can include such varieties as Douglas Furs in the western region of the United States or Black Stinkwood in South Africa. The amount and type of fuel on the floor of a forest also has an impact on the rate at which fire spreads. In addition to weather conditions, moisture content of the fuel and the ground (unless the fire is strictly a crown fire) will either hamper or increase the rate at which fire spreads. The most significant topographical feature that would affect the rate of spread is the slope of the forest terrain. Given certain topological features of the forest floor, the rate of the spread of fire can double due to an increase in slope and in some cases quadruple (Luke and Mcarthur, 1978). Future work also includes studying proper intervention strategies for the rate γ. Incorporating any or all of these factors into a simulation should be included in future studies in order to capture a closer representation of the events surrounding a forest fire and how the spread and the rate of the spread differs in the presence and absence of biologically realistic properties of the forest.

5 Appendix A: Lemmas and Proofs

Lemma 1. *If $a\alpha_B + b\alpha_T < 2(\beta+\gamma)$ and $\alpha_B\alpha_T > \alpha_D\alpha_U$, then FFE is locally asymptotically stable.*

Proof. In order for an equilibrium point to be stable we need $\tau < 0$ and $\Delta > 0$. If $\alpha_B\alpha_T > \alpha_D\alpha_U$ and $a\alpha_B + b\alpha_T < \beta + \gamma$ then $\Delta > 0$. This can be seen by rewriting the determinant as

$$\Delta = ab(\alpha_B\alpha_T - \alpha_D\alpha_U) + (\beta+\gamma)((\beta+\gamma) - (a\alpha_B + b\alpha_T)).$$

However, by these two conditions we are not guaranteed that the trace is negative. Merely by imposing $a\alpha_B + b\alpha_T < 2(\beta+\gamma)$, which does not contradict previous assertions, we have that $\tau < 0$ completing the proof. $\square$

Lemma 2. $c > 0 \Rightarrow b < 0$

Proof. We can start with the following inequality

$$8(\beta+\gamma)^2 + \alpha_B\beta > 0$$

Then this implies the following inequalities

$$\begin{aligned}
12(\beta+\gamma)^2 + \alpha_B(\beta+\gamma) &> \alpha_B\gamma + 4(\beta+\gamma)^2 \Leftrightarrow \\
12(\beta+\gamma) + \alpha_B &> \frac{\alpha_B\gamma + 4(\beta+\gamma)^2}{(\beta+\gamma)} \Leftrightarrow \\
\frac{\alpha_B + 12(\beta+\gamma)}{3\alpha_B} &> \frac{\alpha_B\gamma + 4(\beta+\gamma)^2}{3\alpha_B(\beta+\gamma)}
\end{aligned}$$

So, if $c > 0$ then $\frac{\alpha_B\gamma+4(\beta+\gamma)^2}{3\alpha_B(\beta+\gamma)} > \frac{P[1_B 1_B]^*}{P[1_B]^*}$, then by the last inequality $\frac{\alpha_B+12(\beta+\gamma)}{3\alpha_B} > \frac{P[1_B 1_B]^*}{P[1_B]^*}$ too, i.e. $b < 0$. So in original terms

$$Det(A) > 0 \text{ and } Tr(A) < 0$$

Lemma 3. *Given b and c as before, then $b^2 \nless 4c$.*

Proof. Defining $p := \frac{P[1_B 1_B]^*}{P[1_B]^*}$ and if we suppose that $b^2 < 4c$ then

$$\begin{aligned}
\left(\frac{3\alpha_B p}{4} - 3(\beta+\gamma) - \frac{\alpha_B}{4}\right)^2 &< 4\left[2(\beta+\gamma)\left(\frac{\alpha_B}{4} - \frac{3\alpha_B p}{4} + \beta + \gamma\right) - \frac{\alpha_B\beta}{2}\right] \\
\left(\frac{\alpha_B}{4}(3p-1)^2 - 3(\beta+\gamma)\right)^2 &< 2(\beta+\gamma)(-\alpha_B(3p-1) + 4(\beta+\gamma)) - 2\beta\gamma \\
\frac{\alpha_B^2}{16}(3p-1)^2 - \frac{3\alpha_B}{2}(3p-1)(\beta+\gamma) + 9(\beta+\gamma)^2 &< -2\alpha_B(3p-1)(\beta+\gamma) + 8(\beta+\gamma)^2 - 2\beta\gamma \\
\frac{\alpha_B^2}{16}(3p-1)^2 + \frac{\alpha_B}{2}(3p-1)(\beta+\gamma) + (\beta+\gamma)^2 + 2\beta\gamma &< 0
\end{aligned}$$

Now, if we solve for the discriminant of the last polynomial in $3p-1$ we can see that

$$\begin{aligned}\left(\frac{\alpha_B}{2}(\beta+\gamma)\right)^2 - 4\frac{\alpha_B^2}{16}\left((\beta+\gamma)^2+2\beta\gamma\right) &= \frac{\alpha_B^2}{4}(\beta+\gamma)^2 - \frac{\alpha_B^2}{4}(\beta+\gamma)^2 - \frac{\alpha_B^2\beta}{2} \\ &= -\frac{\alpha_B^2\beta}{2} < 0\end{aligned}$$

But this implies that

$$3p-1 \in \mathbb{C} \Rightarrow p \in \mathbb{C}$$

And this is a contradiction, because $p = \frac{P[1_B 1_B]^*}{P[1_B]^*} \in \mathbb{R}$, so the case $b^2 < 4c$ is not permissible.

6 Appendix B: Sensitivity Analysis

6.1 Dual Layer Mean Field

We can derive the sensitivity Indexes. Given that $\frac{P}{U} = \frac{2\alpha_B}{F_{BB}+F_{TT}+\sqrt{(F_{BB}-F_{TT})^2+4F_{BT}F_{TB}}}$ we have

$$\begin{aligned}
S_{\alpha_B} &= \frac{P}{U}\left(\frac{P[1_B]^*}{(\gamma+\beta)}\left(\frac{1}{2}+\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)\right) \\
S_{\alpha_U} &= \frac{P}{U}\left(\frac{P[1_T]^*}{(\gamma+\beta)}\frac{F_{BT}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right) \\
S_{\alpha_D} &= \frac{P}{U}\left(\frac{P[1_B]^*}{(\gamma+\beta)}\frac{F_{TB}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right) \\
S_{\alpha_T} &= \frac{P}{U}\left(\frac{P[1_T]^*}{(\gamma+\beta)}\left(\frac{1}{2}-\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)\right) \\
S_{P[1B]^*} &= \frac{P}{U}\left(\frac{\alpha_B}{(\gamma+\beta)}\left(\frac{1}{2}+\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)+\frac{\alpha_D}{(\gamma+\beta)}\frac{F_{TB}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right) \\
S_{P[1T]^*} &= \frac{P}{U}\left(\frac{\alpha_U}{(\gamma+\beta)}\frac{F_{BT}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}+\frac{\alpha_T}{(\gamma+\beta)}\left(\frac{1}{2}-\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)\right) \\
S_\beta &= \frac{P}{U}(\frac{-\alpha_B P[1_B]^*}{(\gamma+\beta)^2}\left(\frac{1}{2}+\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)+\frac{-\alpha_D P[1_B]}{(\gamma+\beta)^2}\frac{F_{TB}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}} \\
&+ \frac{-\alpha_U P[1_T]^*}{(\gamma+\beta)^2}\frac{F_{BT}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}+\frac{-\alpha_T P[1T]*}{(\gamma+\beta)^2}\left(\frac{1}{2}-\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)) \\
S_\gamma &= \frac{P}{U}(\frac{-\alpha_B P[1_B]^*}{(\gamma+\beta)^2}\left(\frac{1}{2}+\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)+\frac{-\alpha_D P[1_B]}{(\gamma+\beta)^2}\frac{F_{TB}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}} \\
&+ \frac{-\alpha_U P[1_T]^*}{(\gamma+\beta)^2}\frac{F_{BT}}{((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}+\frac{-\alpha_T P[1T]^*}{(\gamma+\beta)^2}\left(\frac{1}{2}-\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right))
\end{aligned}$$

Given:

$$S_{\alpha_B} = \frac{P}{U}\frac{P[1_B]^*}{(\gamma+\beta)}\left(\frac{1}{2}+\frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}}\right)$$

S_{α_B} is negative when

$$\begin{aligned}\frac{-(F_{BB}-F_{TT})}{2((F_{BB}-F_{TT})^2+4F_{BT}F_{TB})^{1/2}} &< \frac{1}{2} \Leftrightarrow\\ \left((F_{BB}-F_{TT})^2+4F_{BT}F_{TB}\right)^{1/2} &> -(F_{BB}-F_{TT}) \Leftrightarrow\\ \left((F_{BB}-F_{TT})^2+4F_{BT}F_{TB}\right)^{1/2} &> (F_{TT}-F_{BB})\end{aligned}$$

Since we know that $4F_{BT}F_{TB} > 0$, and we also know that:

$$\sqrt{(F_{BB}-F_{TT})^2} = |F_{BB}-F_{TT}| = |F_{TT}-F_{BB}| \geq F_{TT}-F_{BB}$$

Then we may conclude that S_{α_B} is positive, which demonstrates S_{α_B} has a directly proportional relationship with the system. It is clear that S_{α_U} and S_{α_D} are positive since each composition of the expression only involves terms that are greater than zero. Therefore S_{α_U} and S_{α_D} have directly proportional relationships with the system. For S_{α_T} we have a similar argument to α_B, in order to conclude that it is always positive. In order to determine if it is positive we must show the inequality holds:

$$\begin{aligned}\frac{1}{2} &> \frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+F_{BT}F_{TB})^{1/2}} \Leftrightarrow\\ \left((F_{BB}-F_{TT})^2+4F_{BT}F_{TB}\right)^{1/2} &> F_{BB}-F_{TT}\end{aligned}$$

Since we know that $4F_{BT}F_{TB} > 0$, and we also know that:

$$\sqrt{(F_{BB}-F_{TT})^2} = |F_{BB}-F_{TT}| \geq F_{TT}-F_{BB}$$

Therefore

$$\frac{1}{2} > \frac{F_{BB}-F_{TT}}{2((F_{BB}-F_{TT})^2+F_{BT}F_{TB})^{1/2}}$$

It can be observed that $S_{P[1_B]^*}$ and $S_{P[1_T]^*}$ are positive equations since they have terms we have just showed must be positive. So, we may conclude that $S_{P[1_B]^*}$ and $S_{P[1_T]^*}$ have a directly proportional relationship. As we can see S_β and S_γ are composed of a sum of terms that we have showed are positive, multiplied by terms that are always negative. Therefore it is clear that we can conclude that S_β and S_γ are negative, which implies that they have a inversely proportional relationship.

6.2 Single Layer PA model

Recalling that $\mathfrak{F}_0 = \frac{\alpha_B}{P[1_B]^*}\frac{3\gamma P[1_B 1_B]^* + 4\beta P[1_B]^*}{(\beta+\gamma)(4(\beta+\gamma)+\alpha_B)}$ we are able to do a forward sensitivity analysis. Here, we present the results of the operations to determine the sensitivity of $\mathfrak{F}_0$ with respect to changes in the model parameters and give interpretations.

Sensitivity index of α_B

For this parameter obtained the following result:

$$S_{\alpha_B} = 1 - \frac{\alpha_B^2}{4(\beta+\gamma)+\alpha_B}$$

And for the case when $S_{\alpha_B} > 0$ we obtained the quadratic equation

$$\alpha_B^2 - \alpha_B - 4(\beta+\gamma) < 0$$

Then as the discriminant of $\sqrt{1+16(\beta+\gamma)}$ is greater than 1 we only have the positive solution

$$\alpha_B = \frac{1}{2}\left(1 \pm \sqrt{1+16(\beta+\gamma)}\right)$$

So, we have that

$$\text{if } \alpha_B \in \left(0, \frac{1}{2}[1+\sqrt{1+16(\beta+\gamma)}]\right) \Rightarrow \; S_{\alpha_B} > 0$$

Since under this condition S_{α_B} is positive we can say that it is directly proportional, as we increase the value of α_B increases the value of $\mathfrak{F}_o$ i.e. and consequently the spread of the fire increases.

Sensitivity index of β

In this case we have that the value of the sensibility is given by

$$S_\beta = \frac{\beta(4\alpha_B - [8(\beta+\gamma)+\alpha_B]\mathfrak{F}_o)}{K\mathfrak{F}_o}$$

where

$$K := (\beta+\gamma)(4(\beta+\gamma)+\alpha_B)$$

We'd like to know when $S_\beta > 0$ and we found:

$$\beta^2 + P\beta + Q < 0 \tag{20}$$

where

$$\begin{aligned} P: &= \frac{3\gamma P[1_B 1_B]^*}{2P[1_B]^*} \\ Q: &= \frac{1}{16P[1_B]^*}[(8\gamma+\alpha_B)(3\gamma P[1_B 1_B]^*) - 4P[1_B]^*\gamma(4\gamma+\alpha_B)] \end{aligned}$$

The solution to Equation (**??**) is

$$\beta = \frac{1}{2}[-P \pm \sqrt{P^2 - 4Q}]$$

Now as $\beta \in \mathbb{R}$ we need to have $P^2 - 4Q \geq 0$ and with this condition we have that

$$\beta = \frac{1}{2}[-P - \sqrt{P^2 - 4Q}] = \frac{-1}{2}[P + \sqrt{P^2 - 4Q}] < 0$$

Because β is greater than zero, then we going to work with the second root. We can see that

$$\beta > 0 \Rightarrow \; P < \sqrt{P^2 - 4Q} \; \Rightarrow \; Q \leq 0 \; \Rightarrow \; P^2 - 4Q \geq 0$$

So the discriminant always is non-negative and the second solution of (**??**) is positive, then

$$\text{if } \beta \in \left(0, \frac{1}{2}[-P + \sqrt{P^2 - 4Q}]\right) \Rightarrow \; S_\beta > 0$$

With this conditions when we increase the parameter β there are an incremental change in our fire ignition number.

Sensitivity index of γ

For this last parameter the value of its sensitivity its given by

$$S_\gamma = \frac{\gamma[3P[1_B 1_B]^*\alpha_B - P[1_B]^*\mathfrak{F}_\mathfrak{o}(8(\beta+\gamma)+\alpha_B)]}{KP[1_B]^*\mathfrak{F}_\mathfrak{o}}$$

where K is defined as above. Now proceeding as before, we found the following relation

$$\text{if } \gamma \in \left(\frac{1}{2}[M - \sqrt{M^2 + 4N}], \frac{1}{2}[M + \sqrt{M^2 + 4N}]\right) \bigcap \mathbb{R}^+ \; \Rightarrow \; S_\gamma > 0$$

where

$$
\begin{aligned}
M: &= \frac{8\beta P[1_B]^*}{3P[1_B1_B]^*} \\
N: &= \beta[\frac{1}{4}(4\beta+\alpha_B) - \frac{P[1_B]^*}{3P[1_B1_B]^*}(8\beta+\alpha_B)]
\end{aligned}
$$

While this condition holds, there will be a directly proportion relationship between the change of γ and the proportion of fire.

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