Prisoner Reform Programs, and their Impact on Recidivism

Wilson E. Alvarez[1], Itelhomme Fene[2], Kimberly Gutstein[3], Brenda Martinez[4], Diego Chowell[5] Anuj Mubayi[6], Luis Melara[7]

[1] Department of Mathematics, University of Puerto Rico, Mayaguez, PR, USA.
[2] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA, USA.
[3] Department of Mathematics, Humboldt State University, Arcata, CA, USA.
[4] Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA, USA.
[5] Department of Applied Mathematics for the Life and Social Sciences, Arizona State University, Tempe, CA, USA.
[6] Department of Mathematics, Northeastern Illinois University, Chicago, IL, USA.
[7] Department of Mathematics, Shippensburg University, Shippensburg, PA, USA.

Abstract

The California prison system has a high percentage of people who return to prison within a three year period after release. A mathematical model is formulated to study the effectiveness of Reentry Court programs for first time offending parolees designed to reduce the prison return rates when implemented alone or in conjugation with an in prison educational program. Parolees who participated in both in/out of prison programs are referred to as an ideal class in the model. Stability analysis and numerical simulations were carried out to study the impact of the programs. The results show that the reentry program reduces the recidivism rate more than the Basic Educational program within the prison system, but only when social influence of criminals is low outside of prison. However, for populations with high rates of social influences, incarceration rates should be large in order to get the same impact of the reentry program.

1 Introduction

The prison system as we know it today has evolved over the past centuries. The idea of rehabilitation for prisoners was a more recent addition introduced in the 18th century, [2]. Today's prison system serves four purposes: retribution, incapacitation, deterrence and rehabilitation, [1]. Many inmates are eventually released back into society and more than half return to prison more than once. Evidence shows that in recent years there were 760 prisoners per 100,000 individuals per year giving the U.S. the highest rate of incarceration in comparison to the rest of the world, [3]. We need to consider whether the prison reforms within it actually discourage people from committing crimes given that we have high rates of recidivism, [3]. In this report, we seek to gain insight using a mathematical model of the effectiveness of reform programs.

Although California populates less than 5% of the world's population, it is known for having thirty-three prisons with the majority of them being overcrowded, [4]. In 2011, the U.S. Supreme court ruled that California decrease its inmate population by 34,000 in two years, because overcrowded prisons yield poor conditions and violate prisoner's constitutional rights, [5]. In addition, overcrowding also creates unrest and violence which can become dangerous when guards are outnumbered by hundreds of inmates, [6]. Sixty-five percent of released inmates go back to prison within a three year period, [5]. It is in society's best interest that inmates are reformed to prevent them from returning to prison after their release. In California, the average annual cost to house inmates is $45,006 per person, [7]. In [8] it was shown that a reduction of only five percent of the recidivism rate will save $500 million in capital costs per year.

One of the goals of the prison system is to reform criminals into law abiding citizens. Prisons have made drastic changes in management to find a more effective reform system because as society changes, so will prison's objectives, [13]. High recidivism rates suggest that prisoners have been going through a less effective system, one that should be fixed for the welfare of the state. Studies show that lack of education suggests high recidivism, [1]. We focus on two types of educational programs; the first is called the Basic Academic Education Program and is an in-prison type of program and the second is an outside prison program called Reentry Court Program. This program helps a released inmate find a job, since employment keeps parolees out of prison. We propose a mathematical approach to evaluate the impact of the two prison reform programs.

The goal of this mathematical study is to identify mechanisms that may reduce rates of recidivism. Our model has several features in common to the MTBI technical report, "Dynamical Interpretation of the Three Strikes Law," [10]. Both models focus on criminals who commit violent crimes, such as homicide, arson, robbery, rape, motor vehicle theft and aggravated assault, to name a few. A compartmental and deterministic model is used to simulate and analyze criminal activity as an infectious social disease. We consider the male population at high risk of becoming criminals in the entire state of California. For simplicity, we stratified the population into susceptibles, criminals that are free, prisoners, and released prisoners. We assume that new criminals are only produced through social influence on the susceptible population, i.e. a population of non-criminals and released prisoners.

Unlike the model in [10], we look at reform programs. We consider a prison system that has external and internal programs to reduce recidivism. The program we focus on within the prisons is a Basic Educational program, where criminals can take classes that are similar to K-12 and can earn their General Education Diploma (GED), which is equivalent to a high school diploma. The goal of this program is to prepare the inmates for success outside of prison and to enhance the rehabilitative aspects of prison. Educational programs offered inside prisons are typically provided and managed by state prison systems in which they reside. Funding for the programs are granted through state or federal correctional department budgets. The outside program is a reentry program that helps ex-prisoners integrate back into society and helps them from returning to prison a second time, [17]. Reentry Court programs are a new trend in California in attempts to reduce recidivism. They are called Reentry Court programs because they require judicial monitoring of parolees to promote public safety, [21]. Only ex-offenders are admitted into these programs. Both programs are optional for the prisoners, [15].

It is believed that prisoners who complete both programs are less likely to return to prison, [4]. We are addressing whether the Basic Academic Education Program is more effective in reducing recidivism or whether the Reentry Court Program has a greater impact in reducing recidivism. Figure 1 describes the flow of men through the prison system and educational programs. We model this from an epidemiological perspective where the disease is "crime". The compartment S corresponds to individuals in the susceptible class who have never committed a crime. The C class corresponds to those individuals who have committed a crime but have never been convicted for it or served prison time. The I compartment represents those individuals who are in prison for the first time which acts as a quarantine while inmates go through the recovery process.

The model contains four classes that leave the incarcerated class (I): H_1 class corresponds to both the Basic Education Program and the Reentry Court Program, individuals who only completed the Basic Academic Program are in A_1 and men who did not complete the in-prison program (Basic Academic Education Program) but did finish the outside programs (Re-Entry Court Program) are in H_2. Finally, we included the group of people who did not complete the inside program or the outside program (A_2). Therefore the A_2 class is considered trivial, since it will yield the largest recidivism. To address this question, we study three simplified models (H_1, H_2, A_1).

The model in Figure 1 was simplified into three mathematically equivalent models. Thus, we perform mathematical analysis on the simplified model involving A_1 only. We let H_1 class be the ideal group and expect it to yield the lowest recidivism rate. We want to compare the effectiveness of the classes who complete only one program (H_2 and A_1). The model incorporates social influence of criminals on susceptibles and the effectiveness of recidivism related interaction programs, [15]. The results of this research may yield insight into the reduction of recidivism rates in California.

This article is organized as follows: In section 2, we provide and discuss the general framework of the model. In section 3, we analyze the simplified model. The values of numerical simulations are shown in section 4. The implications of our results and analysis, includes limitations of the models in section 5 and also suggest further research.

2 Data Sources

In this section we describe the data sources used to justify model parameters that we use as our estimates. We only consider the male population who is 18 or older. The US Census Bureau data is used to estimate the 18 or older total population of U.S.

and California. The number of individuals who complete the education program within prison are obtained from the estimates on the California Department of Corrections and Rehabilitation (Office of Correctional Education) [8]. The total number of male inmates incarcerated per month from 2006-2010 was also collected and used, [19]. The data from the 2011 prison evaluation report is used to compute the release rate. Data for the reentry program is obtained from [5].

3 Model

Our model considers males 18 years and older in the state of California. A compartmental model is shown in Figure 1. Susceptible individuals (S) in our model can become criminals (C) under the social influence of criminals outside of prisons (C, A_1, H_1, A_2, H_2). The criminals are caught and imprisoned (I) at the per capita rate σ per month. Released individuals can transition into one of four compartments based on the type of reform programs that they have completed or will be completing. The recruitment rate, Λ, represents the number of males who turn 18 per month in the state of California; μ represents the per capita exit rate for each class; θ_1 will denote the per capita re-incarceration rate for the first-time offenders who completed the educational program (inside prison), while θ_2 will represent the per capita re-incarceration rate for not completing the educational program. The parameter p captures the proportion of released individuals that complete the Basic Education program and q correlates with the proportion of released individuals that complete a Reentry Court program. The parameter σ represents the per capita incarceration rate. We define β as a social influence parameter related to individuals in the criminal class which is known as a transmission rate in "regular" epidemiological models. In order to quantify social influence related parameters we use the data from 2006 through 2010. We let $\beta_1 = \beta\epsilon_1$, where $0 < \epsilon_1 < 1$ is a weight. β_1 corresponds to the social influence parameter of the A_1 class on the S class. β_2, β_3, and β_4 represent the social influence parameter of A_2, H_2, H_1, respectively, on the susceptible class. We assume $\beta > \beta_2 > \beta_1 > \beta_3 > \beta_4$ to determine values of $\epsilon_1 - \epsilon_4$.

First, we focused on three simplified one-intervention-compartment model and an example model is shown in Figure 2. Each model focuses on a population of first time parolees: H_1, H_2, and A_1.

Class	Description
S	Population that is susceptible to becoming criminals
C	Individuals that commit crimes and are not imprisoned
I	Imprisoned criminal population
A_1	Released inmates that completed the inside program, but did not complete or were not involved in outside program
A_2	Released inmates that did not complete or were not involved the inside program within prison and also did not complete or were not involved in the outside program
H_1	Released inmates that completed the inside program, and completed or were involved in the outside program (control group)
H_2	Released inmates that did not complete or were not involved in the inside program, but completed or were involved the outside program
R	Released inmates that go back to prison a second time irrespectively of their past experience with the reform programs

Table 1: Compartmental classes and interpretations

Parameters	Description
N	Starting population
Λ	Rate of entry of individuals into core-group population as susceptibles
σ	Per capita incarceration rate
γ	Per capita prison release rate
μ	Per capita mortality rate
q	Proportion of released individuals that were involved in outside program
p	Proportion of released individuals that completed the inside program
θ_1	Per capita reincarceration rate for those who completed the inside program
θ_2	Per capita reincarceration rate for not completing the inside program
β	Social influence parameter related to individuals in the C class
$\beta_i = \epsilon_i \beta$	Social influence parameter related to individuals in the A_1, A_2, H_1, H_2 class where ϵ_i is the reduction in the social influence of A_1, A_2, H_1, H_2 class individuals as compared to C class individuals
ϕ	Proportion of unsuccess rate of the reentry program
α	Proportion of individuals that go back to prison

Table 2: Parameters.

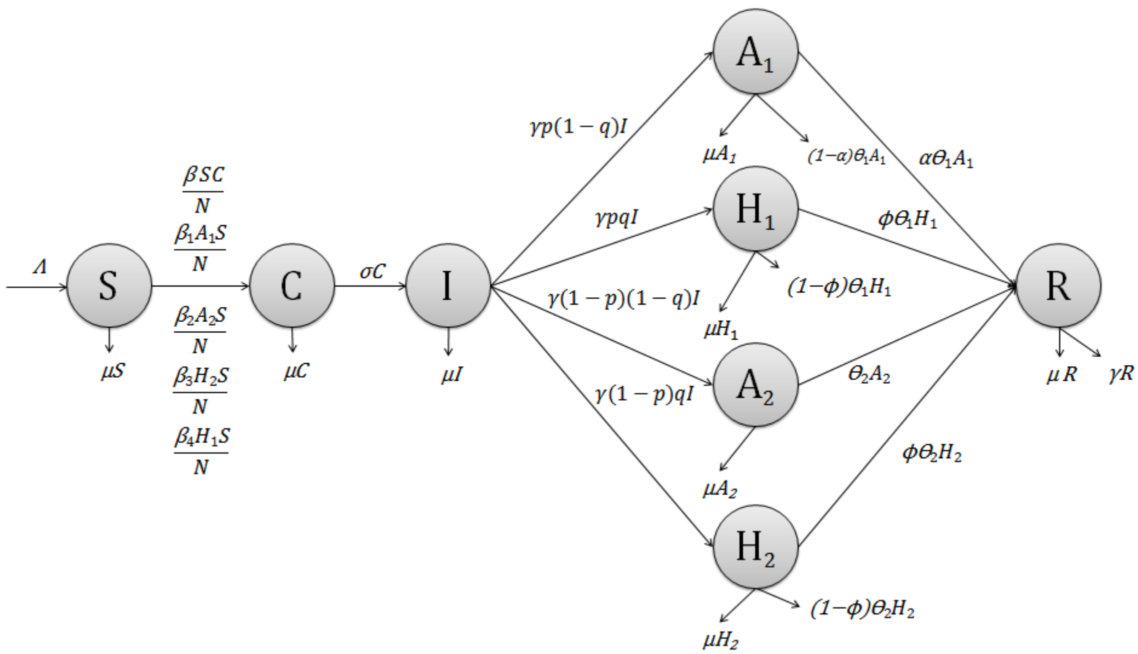

Figure 1: Flow diagram of the general model.

3.1 Simplified Model Involving A_1 Compartment

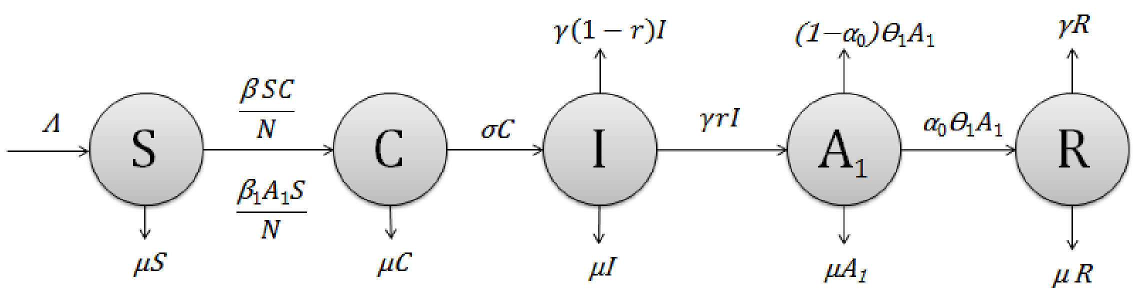

Figure 2: Simplified A_1 model

The system of ODE's corresponding to Figure 2 is:

$$
\begin{array}{rcl}
\frac{dS}{dt} & = & \Lambda - \frac{\beta CS}{N} - \frac{\beta_1 A_1 S}{N} - \mu S \\
\frac{dC}{dt} & = & \frac{\beta CS}{N} + \frac{\beta_1 A_1 S}{N} - \mu C - \sigma C \\
\frac{dI}{dt} & = & \sigma C - \gamma I - \mu I \\
\frac{dA_1}{dt} & = & \gamma r I - \theta_1 A_1 - \mu A_1 \\
\frac{dR}{dt} & = & \alpha_0 \theta_1 A_1 - \gamma R - \mu R
\end{array} \tag{1}
$$

where, $r = p(1-q)$

In this model we only consider the S, C, I, A_1, and R classes. In our susceptible population (S) we have our inflow of people turning 18 (Λ), and three outflows: μS symbolizes the susceptible individuals leaving our system due to mortality, $\frac{\beta SC}{N}$ and $\frac{\beta_1 A_1 S}{N}$ which represents the amount of susceptible individuals that turn into criminals over time. $\frac{\beta SC}{N}$ and $\frac{\beta_1 A_1 S}{N}$ go into C, the criminal class, and has two outflows: μC denotes the criminals leaving our system due to mortality, and σC which represents the number of never imprisoned criminals that are incarcerated over time. The rate σC goes into our I class, which contains the first-time prisoners. This class has three outflows: μI re presents first-time prisoners who leave the system due to mortality, $\gamma(1-r)I$ symbolizes the number of first-time released inmates who enter different programs other than A_1, and $\gamma r I$ is the number of first-time released inmates who go into the A_1 class over time. This last class has three outflows: μA_1 which represents the number of first-time released inmates who went through an in-prison program but are leaving the system due to mortality, $(1-\alpha_0)\theta_1 A_1$ stands for the number of completely recovered first-time released inmates over time, and $\alpha_0 \theta_1 A_1$ represents the number of second-time offenders (i.e. first-time released inmates that fall into recidivism) over time. $\alpha_0 \theta_1 A_1$ goes into R, which is the class that contains second-time offenders. This class has two outflows: γR stands for the number of second-time release inmates over time, and μR denotes the second-time offenders that leave our system due to mortality.

4 Analysis

The mathematical analysis for the simplified model A_1 is present. The analysis for the simplified models H_1 and H_2 will be similar because only parameters change. The equilibrium points obtained from this model are obtained by finding roots of (1), which we found using Maple 16. The crime-free equilibrium (CFE) is $S^* = N$, $C^* = 0$, $I^* = 0$,

$A_1^* = 0$, $R^* = 0$, and the recidivism equilibrium point is given by

$$\begin{aligned}
S^* &= \frac{N s_1 s_3 s_2}{\beta s_1 s_2 + \beta_1 \sigma\gamma r} \\
C^* &= \frac{s_1 s_2 \left(-\mu s_3 N + \Lambda\beta\right) + \Lambda\sigma\gamma r\beta_1}{s_3 \left(\beta s_1 s_2 + \beta_1 \sigma\gamma r\right)} \\
I^* &= \frac{\left(s_1 s_2 \left(-\mu s_3 N + \Lambda\beta\right) + \Lambda\sigma\gamma r\beta_1\right)\sigma}{s_3 s_2 \left(\beta s_1 s_2 + \beta_1 \sigma\gamma r\right)} \\
{A_1}^* &= \frac{\sigma\gamma r \left(s_1 s_2 \left(-\mu s_3 N + \Lambda\beta\right) + \Lambda\sigma\gamma r\beta_1\right)}{s_1 s_3 s_2 \left(\beta s_1 s_2 + \beta_1 \sigma\gamma r\right)} \\
R^* &= \frac{\alpha_0 \theta_1 \sigma\gamma r \left(s_1 s_2 \left(-\mu s_3 N + \Lambda\beta\right) + \Lambda\sigma\gamma r\beta_1\right)}{s_1 s_3 {s_2}^2 \left(\beta s_1 s_2 + \beta_1 \sigma\gamma r\right)}
\end{aligned}$$

where $s_1 = \mu + \theta_1$, $s_2 = \mu + \gamma$, and $s_3 = \mu + \sigma$.

The crime reproductive number, $\mathcal{R}_c$, is computed using the next generation operator method [17], with the following vectors:

$$\mathcal{F} = \begin{bmatrix} \frac{\beta SC}{N} + \frac{\beta_1 A_1 S}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} (\mu+\sigma)C \\ -\sigma C + (\gamma+\mu)I \\ -\gamma r I + \theta_1 A_1 + \mu A_1 \\ -\alpha_0\theta_1 A_1 + (\gamma+\mu)R \end{bmatrix},$$

where $\mathcal{F}$ is the vector of rates of appearance of new criminals in each compartment, and $\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$ is the vector of transferring rates of individuals into and out of the compartments. The crime reproductive number is therefore given by

$$\mathcal{R}_c = \frac{\beta}{\mu+\sigma} + \frac{\beta_1 \gamma r \sigma}{(\gamma+\mu)(\mu+\sigma)(\mu+\theta_1)}. \tag{2}$$

The details of the computations are shown explicitly in Appendix A.

Theorem 1. *If $\mathcal{R}_c < 1$, then the Crime-Free Equilibrium $(N, 0, 0, 0, 0)$ is locally asymptotically stable.*

Proof

The Jacobian of the system of differential equations (1) is given by

$$\begin{bmatrix}
-\frac{\beta C^*}{N} - \frac{\beta_1 A_1^*}{N} - \mu & -\frac{\beta S^*}{N} & 0 & -\frac{\beta_1 S^*}{N} & 0 \\
\frac{\beta C^*}{N} + \frac{\beta_1 A_1^*}{N} & \frac{\beta S^*}{N} - \sigma - \mu & 0 & \frac{\beta_1 S^*}{N} & 0 \\
0 & \sigma & -\gamma-\mu & 0 & 0 \\
0 & 0 & \gamma r & -\mu-\theta_1 & 0 \\
0 & 0 & 0 & \alpha\theta_1 & -\gamma-\mu
\end{bmatrix}.$$

Evaluating at the CFE $(N, 0, 0, 0, 0)$ yields

$$\begin{bmatrix} -\mu & -\beta & 0 & -\beta_1 & 0 \\ 0 & \beta-\mu-\sigma & 0 & \beta_1 & 0 \\ 0 & \sigma & -\gamma-\mu & 0 & 0 \\ 0 & 0 & \gamma r & -\mu-\theta_1 & 0 \\ 0 & 0 & 0 & \alpha\theta_1 & -\gamma-\mu \end{bmatrix}. \tag{3}$$

Note that $\lambda_1 = -\mu$, and $\lambda_2 = -\gamma - \mu$ are two negative eigenvalues since all parameters are greater than zero. Next we show that the remaining three eigenvalues are negative. We start by eliminating the columns and rows of (3) corresponding to λ_1 and λ_2. This reduces the Jacobian to the following matrix

$$\begin{bmatrix} \beta-\mu-\sigma & 0 & \beta_1 \\ \sigma & -\gamma-\mu & 0 \\ 0 & \gamma r & -\mu-\theta_1 \end{bmatrix}. \tag{4}$$

This submatrix has the following characteristic polynomial:

$$\begin{aligned} &\lambda^3 + (\gamma+\theta_1-\beta+\sigma+3\mu)\,\lambda^2 + (-\beta\gamma+\sigma\gamma+2\gamma\mu+\sigma\theta_1+3\mu^2-2\beta\mu \\ &+2\mu\sigma-\beta\theta_1+2\mu\theta_1+\gamma\theta_1)\lambda - \beta_1\sigma\gamma r - \beta\gamma\theta_1 + \sigma\gamma\theta_1 + \mu\gamma\theta_1 - \beta\mu\theta_1 + \sigma\mu\theta_1 \\ &+\mu^2\theta_1 - \beta\gamma\mu + \sigma\gamma\mu + \mu^2\gamma - \beta\mu^2 + \sigma\mu^2 + \mu^3. \end{aligned}$$

Let

$$\begin{aligned} a_1 &= \gamma+\theta_1-\beta+\sigma+3\mu, \\ a_2 &= -\beta\gamma+\sigma\gamma+2\gamma\mu+\sigma\theta_1+3\mu^2-2\beta\mu+2\mu\sigma-\beta\theta_1+2\mu\theta_1+\gamma\theta_1, \\ a_3 &= \beta_1\sigma\gamma r - \beta\gamma\theta_1 + \sigma\gamma\theta_1 + \mu\gamma\theta_1 - \beta\mu\theta_1 + \sigma\mu\theta_1 + \mu^2\theta_1 - \beta\gamma\mu + \sigma\gamma\mu + \mu^2\gamma - \beta\mu^2 + \sigma\mu^2 + \mu^3. \end{aligned}$$

We show

$$(i.)\ a_1 > 0,\ \ (ii.)\ a_3 > 0, \text{and}\ (iii.)\ a_1a_2 > a_3$$

$(i.)$ Since $\mathcal{R}_c < 1$, then from (2) we have $0 < \frac{\beta}{\mu+\sigma} < \mathcal{R}_c < 1 < 1 + \frac{2\mu+\gamma+\theta_1}{\mu+\sigma}$ which implies

$$\begin{aligned} \frac{\beta}{\mu+\sigma} &< 1 + \frac{2\mu+\gamma+\theta_1}{\mu+\sigma}, \\ \beta &< (\mu+\sigma) + 2\mu + \gamma + \theta_1, \\ 0 &< -\beta + \mu + \sigma + 2\mu + \gamma + \theta_1, \\ 0 &< \gamma + \theta_1 - \beta + \sigma + 3\mu = a_1. \end{aligned}$$

This proves $a_1 > 0$.

$(ii.)$ Note that $\mathcal{R}_c = \frac{\beta_1\sigma\gamma r+\beta(\mu+\sigma)(\mu+\gamma)}{(\mu+\sigma)(\mu+\gamma)(\mu+\theta_1)} < 1$, therefore:

$$\begin{aligned}
\beta_1\sigma\gamma r + \beta(\mu+\theta_1)(\mu+\gamma) &< (\mu+\sigma)(\mu+\gamma)(\mu+\theta_1) \\
0 &< (\mu+\sigma)(\mu+\gamma)(\mu+\theta_1) - \beta_1\sigma\gamma r - \beta(\mu+\theta_1)(\mu+\gamma) \\
0 &< -\gamma\sigma\beta_1 - \mu\gamma\beta + \mu^2\gamma + \mu\gamma\sigma - \mu^2\beta + \mu^3 + \mu^2\sigma \\
&\quad -\theta_1\gamma\beta + \theta_1\gamma\mu + \theta_1\gamma\sigma - \theta_1\mu\beta + \theta_1\mu^2 + \theta_1\mu\sigma = a_3.
\end{aligned}$$

So we have $a_3 > 0$.
$(iii.)$ Let $s_1 = \mu+\sigma$, $s_2 = \mu+\gamma$, $s_3 = \mu+\theta_1$, where $s_1 > 0$, $s_2 > 0$, $s_3 > 0$. Then

$$\begin{aligned}
a_1 &= s_1 + s_2 + s_3 - \beta, \\
a_2 &= (s_2+s_3)(s_1-\beta) + s_2 s_3, \\
a_3 &= s_1 s_2 s_3 - \beta s_2 s_3 - \beta_1 p\sigma\gamma, \\
&= (s_1 s_2 s_3)(1-\mathcal{R}_c).
\end{aligned}$$

To prove $a_1 a_2 > a_3$, we first prove $(s_1 s_2 s_3)(1-\frac{\beta}{s_1}) > a_3$.
Note that $\mathcal{R}_c = \frac{\beta}{s_1} + \frac{\beta_1 r\sigma\gamma}{s_1 s_2 s_3}$.
Then,

$$\begin{aligned}
-\frac{\beta}{s_1} &> -\mathcal{R}_c \\
1-\frac{\beta}{s_1} &> 1-\mathcal{R}_c \\
(s_1 s_2 s_3)(1-\frac{\beta}{s_1}) &> (s_1 s_2 s_3)(1-\mathcal{R}_c) \\
(s_1 s_2 s_3)(1-\frac{\beta}{s_1}) &> a_3.
\end{aligned}$$

Observe $\frac{\beta}{s_1} < \mathcal{R}_c < 1$. Next we show $a_1 a_2 > (s_1 s_2 s_3)(1-\frac{\beta}{s_1})$, which is equivalent to showing $\frac{a_1 a_2}{(s_1 s_2 s_3)(1-\frac{\beta}{s_1})} > 1$. Expanding the left hand side of this last inequality:

$$
\begin{aligned}
\frac{a_1 a_2}{(s_1 s_2 s_3)(1-\frac{\beta}{s_1})} &= a_1 \left(\frac{(s_2+s_3)(s_1-\beta)+s_2 s_3}{(s_1 s_2 s_3)(1-\frac{\beta}{s_1})} \right), \\
&= a_1 \left(\frac{(s_2+s_3)(s_1-\beta)}{(s_2 s_3)(s_1-\beta)} + \frac{s_2 s_3}{(s_1 s_2 s_3)(1-\frac{\beta}{s_1})} \right), \\
&= a_1 \left(\frac{s_2+s_3}{s_2 s_3} + \frac{1}{s_1-\beta} \right), \\
&= a_1 \left(\frac{1}{s_3} + \frac{1}{s_2} + \frac{1}{s_1-\beta} \right), \\
&= (s_1+s_2+s_3-\beta) \left(\frac{1}{s_3} + \frac{1}{s_2} + \frac{1}{s_1-\beta} \right), \\
&= \frac{s_1-\beta+s_2}{s_3} + 1 + \frac{s_1-\beta+s_3}{s_2} + 1 + \frac{s_2+s_3}{s_1-\beta} + 1, \\
&= \frac{s_1-\beta+s_2}{s_3} + \frac{s_1-\beta+s_3}{s_2} + \frac{s_2+s_3}{s_1-\beta} + 3.
\end{aligned}
$$

and since $0 < \frac{\beta}{s_1} < 1$ then
$\frac{s_2+s_3}{s_1-\beta} + \frac{s_1-\beta+s_3}{s_2} + \frac{s_1-\beta+s_2}{s_3} + 3 > 3 > 1$.

Therefore we have $a_1 a_2 > a_3$ as required, which implies that the remaining eigenvalues are negative. Thus, we conclude that the CFE is locally asymptotically stable. □

The stability of recidivism equilibrium was only verified numerically. In our sensitivity analysis we found that the sensitivity indices for $\mathcal{R}_c$ are:

$$
\begin{aligned}
S_\beta &= \frac{\beta(\gamma+\mu)(\mu+\theta_1)}{\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r} \\
S_{\beta_1} &= \frac{\beta_1\sigma\gamma r}{\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r} \\
S_\sigma &= -\frac{\sigma\left(\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1-\beta_1\gamma r\mu\right)}{(\sigma+\mu)(\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r)} \\
S_\mu &= -\frac{\mu\left(\beta(\gamma+\mu)^2(\mu+\theta_1)^2+\sigma\gamma r\left((2\mu+\gamma+\theta_1)\sigma+3\mu^2+(2\gamma+2\theta_1)\mu+\gamma\theta_1\right)\beta_1\right)}{(\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r)(\gamma+\mu)(\sigma+\mu)(\mu+\theta_1)} \\
S_\gamma &= \frac{\gamma\beta_1\sigma r\mu}{(\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r)(\gamma+\mu)} \\
S_{\theta_1} &= -\frac{\theta_1\beta_1\sigma\gamma r}{(\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r)(\mu+\theta_1)} \\
S_r &= \frac{\beta_1\sigma\gamma r}{\beta\gamma\mu+\beta\gamma\theta_1+\beta\mu^2+\beta\mu\theta_1+\beta_1\sigma\gamma r}
\end{aligned}
$$

5 Results

In this section we provide numerical estimates of the model parameters, discuss sensitivity analysis [20] of the crime reproductive number and show numerical simulations of the model.

5.1 Estimation of Parameters

We start by estimating Λ. The percent of female population in the state of California by 2011 is 50.3%, [9], then the percent of male population is 49.7%. The total male and female population from the United States that are 18 and 19 years old is 9,086,089 [9], which implies that the number of people that are only 18 years old in the United States is given by $\frac{9{,}086{,}089}{2}$ people. To find the number of people that are 18 years old in California, use the percentage of people that live in California in comparison to the number of people that live in the Unites States. Use data from [9] we have

$$\frac{\text{Number of California Residents}}{\text{Number of U.S. citizens}} = \frac{37,691,912}{311,591,917}$$

which gives us that the number of people that are 18 years old in California is

$$
\begin{aligned}
\frac{\text{Number of California Residents}}{\text{Number of U.S. citizens}}(\text{U.S. Population}) &= \left(\frac{37,691,912}{311,591,917}\right)\left(\frac{9,086,089}{2}\right) \\
&= 549,552\text{people}
\end{aligned}
$$

Duration of Prison Term (in months)	Proportion of Prison Population In California (unitless)
0 to 6	0.151
6 to 12	0.39
12 to 18	0.165
18 to 24	0.093
24 to 36	0.085
36 to 48	0.038
48 to 60	0.025
60 to 120	0.042
120 to 180	0.009

Table 3: This represents the distribution of incarceration time for prisoners [3].

And since the population of California consists of 49.7% male members, we have that the total number of people that are 18 year old California is given by

$$\begin{aligned}\text{(California 18 year old pop.)(Male Percent)} &= \left(\frac{37691912}{311591917}\right)\left(\frac{9086089}{2}\right)(.497) \\ &= 273127\text{people}\end{aligned}$$

Since Λ represents the number of people that turn 18 each month, we have

$$\frac{(\frac{37691912}{311591917})(\frac{9086089}{2})(.497)}{12} = 22760.62162 \approx 22760 \text{ per month}$$

The release rate γ is computed using the Table 3, [3]. γ is the weighted average of prison terms, which gives us the average time spent in prison.

We use data in table to estimate average,

$$\gamma = 0.052337 \text{ per month.}$$

To calculate μ we assume that the average lifespan of individuals living in California is 70, and subtract 17 years from it because the population is 18 years of age and older. Therefore, individuals in the model live an average of 53 years. This equals $53 * 12 = 636$ months, and we have

$$\mu = \frac{1}{636} = 0.0016 \text{ per month}$$

The Court Reentry Program admitted 656 parolees (males and females). From this group, 83% are males, [5]. To calculate the total male parolee number, we calculate $656 * .83 \approx 544$. The monthly rate: $\frac{544}{12} \approx 45$, that is 45 male parolees participate in the court reentry program every month. We know the total number of releases so we then subtract the number of people doing the reentry program from the total number of releases: $7081 - 45 = 7036$ parolees who did not do the outside program per month. We compute the prgram q:

$$q = \frac{45}{7081} \approx 0.006$$

To find p, we use [6]. From that data we calculated the average number of people who completed the in-prison educational program (Program Completions) in one year. We also calculated the weighted averaged number of people who were released (Total Exits) in one year:

$$\begin{aligned}
\text{Average Completions Per Year} &= 550 + 445 + 456 + 466 + 504 + 355 + 236 + \\
&\quad 314 + 337 + 314 + 399 + 632, \\
&= 5,008.
\end{aligned}$$

$$\begin{aligned}
\text{Average Exits Per Year} &= 2,396 + 2,126 + 5,260 + 2,805 + 1,869 + 2,170 + \\
&\quad 3,001 + 2,536 + 3,329 + 3,159 + 2,894 + 3,551, \\
&= 3,5096.
\end{aligned}$$

To find proportion p, we compute:

$$p = \frac{\text{Average Completions Per Year}}{\text{Average Exits Per Year}} \approx 0.14$$

. Note that p is unitless, we can use this valuein our model to represent the percentage of people who completed the program out of the total prison population.

For the average time spent in the reentry programs, θ_1, there were three different length of programs: six months, twelve months, and eighteen months. We took the average for the length of the program which is twelve months. We then divided one by twelve to get the theta parameter:

$$\frac{1}{\theta_1} = 12$$

$$\Rightarrow \theta_1 = \frac{1}{12}$$

The proportion of recidivism within a six month period after parolees complete the Reentry Court program is 0.23, [5].

Overall 65% of the California's released population goes back to prison within a three yr time period also direct data [5]

Sensitivity index	Value
S_β	0.8336
S_σ	-0.8283
S_μ	-0.0135
S_γ	0.0051
S_{θ_1}	-0.16323
S_p	0.1664
S_{β_1}	0.1664

Table 4: Numerical values of each sensitivity index for parameter values given in Table 5

5.2 Sensitivity Analysis

In this section we perform a sensitivity analysis on the A_1 model which represents the effect the parameters of A_1 have on $\mathcal{R}_c$ through the following equation

$$S_\lambda = \frac{\Delta\mathcal{R}_c}{\mathcal{R}_c} / \frac{\Delta\lambda}{\lambda} = \frac{\lambda}{\mathcal{R}_c} \cdot \frac{\partial\mathcal{R}_c}{\partial\delta},$$

where λ represents each of our parameters.

The index value measures sensitivity of $\mathcal{R}_c$ due to small changes in value of the parameters. If $S_\lambda > 0$ then as λ increases $\mathcal{R}_c$ increases, similarly if λ decreases then so does $\mathcal{R}_c$. If the index value S_λ is negative then λ increases, then $\mathcal{R}_c$ decreases and vice versa.

Using the formula for $\mathcal{R}_c$ given in (2), we also computed the sensitivity indices for $\sigma = 0.15, 0.25, 0.35$ and $\beta = 0.4, 0.5, 0.6, 0.7, 0.8$, where $\beta > \sigma$. The numerical results showed that the $\mathcal{R}_c$ is most sensitive to changes on β for all these cases.

A decrease in β by $\frac{1}{S_\beta} = \frac{1}{0.8154636333} = 1.22\%$ or an increase in σ by $\frac{1}{S_\sigma} = \frac{1}{0.8108604290} = 1.23\%$ will result in a decrease in $\mathcal{R}_c$ by 1%. This means that if the contact rate between criminals and susceptibles is decreased by 1.23%, then the number of new criminals generated by existing criminals will decrease by 1 %. Also, if the incarceration rate of criminals is increased by 1.23%, then then the number of new criminals generated by existing criminals will also decrease by 1%.

5.3 Numerical Results

In this section we discuss our numerical results.

Parameters	Units	Numerical Value	Reference
N	people	13,851,800	[9]
Λ	people $\cdot$ months^{-1}	22760	[9]
σ	months^{-1}	when fixed: 0.3	[11]
γ	months^{-1}	.05	[3]
μ	months^{-1}	$\frac{1}{636}$	assumption
q	$\frac{\text{\# prisoners who completed the program}}{\text{total prisoners released}}$ (unitless)	.006	[5]
p	$\frac{\text{\# prisoners who completed the program}}{\text{total prisoners released}}$ (unitless)	.14	[6]
θ_1	months^{-1}	0.08	[11]
θ_2	months^{-1}	0.1	[11]
β	months^{-1}	0.5	[10]
β_1	months^{-1}	when fixed: $\beta * (.4) = .2$	[14]
β_2	months^{-1}	0.5	[14]
β_3	months^{-1}	0.2	[14]
β_4	months^{-1}	0.15	[14]
ϕ	proportion (unitless)	.23	[5]
α	proportion (unitless)	.65	[5]
$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$	unitless	[0,1]	assumption

Table 5: Classes and Meanings

5.3.1 Simplified Model Containing A_1

From the previous analysis we know that our recidivism equilibrium is:

$$\begin{aligned}
S^* &= \frac{Ns_1s_3s_2}{\beta s_1s_2 + \beta_1\sigma\gamma p} \\
C^* &= \frac{s_1s_2\left(-\mu s_3N + \Lambda\beta\right) + \Lambda\sigma\gamma p\beta_1}{s_3\left(\beta s_1s_2 + \beta_1\sigma\gamma p\right)} \\
I^* &= \frac{\left(s_1s_2\left(-\mu s_3N + \Lambda\beta\right) + \Lambda\sigma\gamma p\beta_1\right)\sigma}{s_3s_2\left(\beta s_1s_2 + \beta_1\sigma\gamma p\right)} \\
{A_1}^* &= \frac{\sigma\gamma p\left(s_1s_2\left(-\mu s_3N + \Lambda\beta\right) + \Lambda\sigma\gamma p\beta_1\right)}{s_1s_3s_2\left(\beta s_1s_2 + \beta_1\sigma\gamma p\right)} \\
R^* &= \frac{\alpha\theta_1\sigma\gamma p\left(s_1s_2\left(-\mu s_3N + \Lambda\beta\right) + \Lambda\sigma\gamma p\beta_1\right)}{s_1s_3{s_2}^2\left(\beta s_1s_2 + \beta_1\sigma\gamma p\right)}
\end{aligned}$$

where $s_1 = \mu + \theta_1$, $s_2 = \mu + \gamma$, and $s_3 = \mu + \sigma$. Using our values from Table 5 we have that $s_1 = 0.08$, $s_2 = 0.05$, and $s_3 = 0.34$, and that our endemic equilibrium is:

$$
\begin{aligned}
S &= 6,964,096.01 \\
C &= 39,161.96031 \\
I &= 227,807.99 \\
A_1 &= 19,548.98 \\
R &= 19,711.10
\end{aligned}
$$

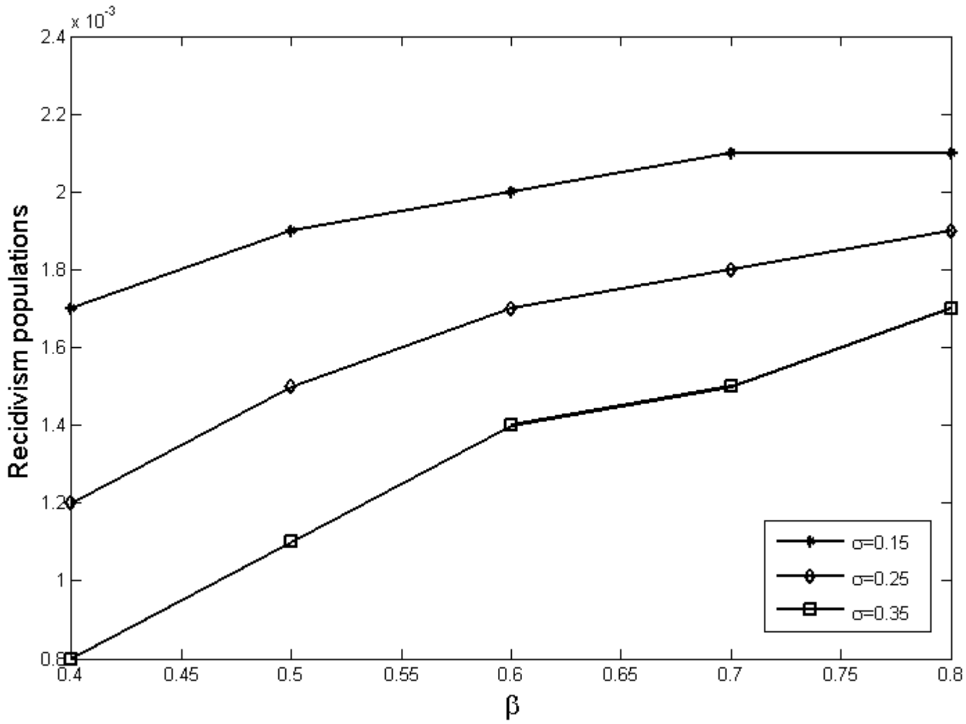

Figure 3: β vs. Recidivism Population

For low social influence β, increase in incarceration rate σ has a higher effect on the size of the recidivism class. For large β and σ, the size of the recidivism class decreases less. We conclude that because the simplified models are mathematically equivalent; this is the case for all programs.

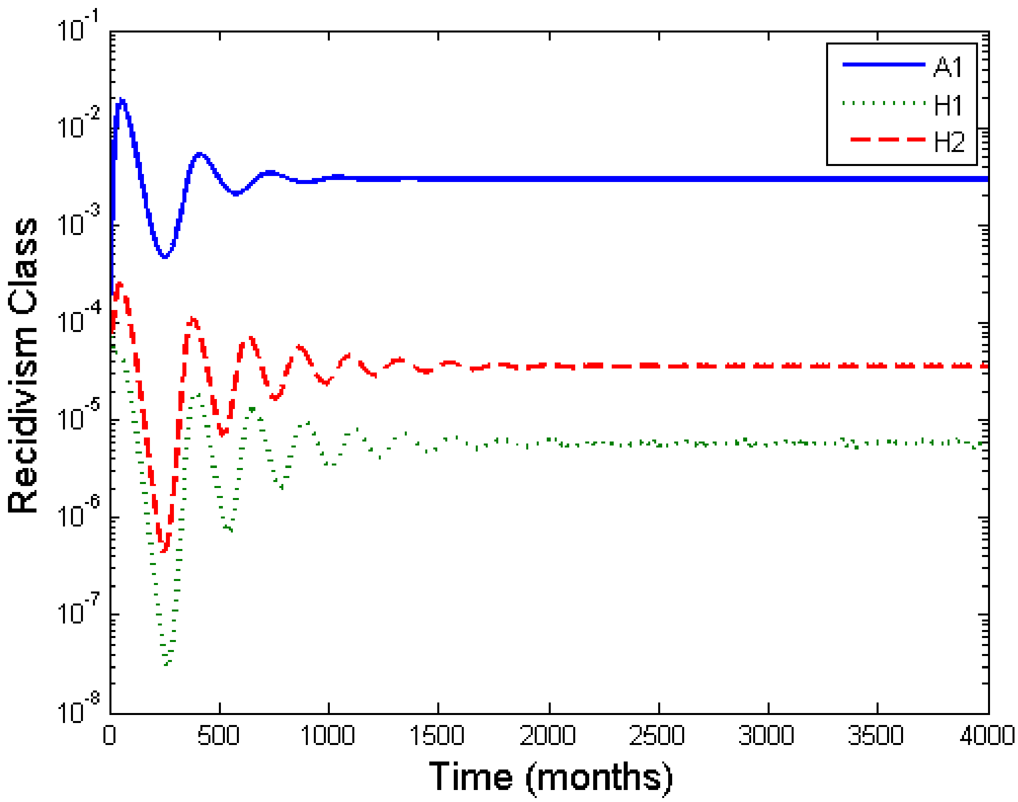

Figure 4: Recidivism Population

Figure 4 shows the proportion of the A_1, H_1, H_2 class going into the recidivism class. Only completing the inside program produces the highest proportion of criminals going back to prison. Completing the outside prison program produces a lower proportion of recidivism and completing both programs produces the lowest recidivism proportion.

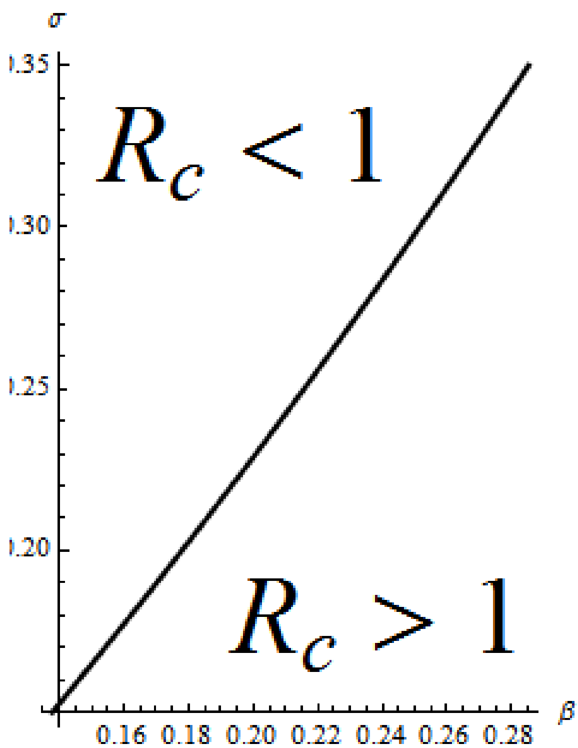

Figure 5: Projection of $\mathcal{R}_c = 1$ onto $\beta\sigma$-plane

Figure 5 shows the impact when β and σ are varied in $\mathcal{R}_c$. When β and σ are on the curve, then $\mathcal{R}_c = 1$. When β and σ are below the curve, then $\mathcal{R}_c > 1$, which means the infection (crime) will continue to spread and therefore will cause an epidemic. When β and σ are above the line, then $\mathcal{R}_c < 1$, which implies the infection will die off.

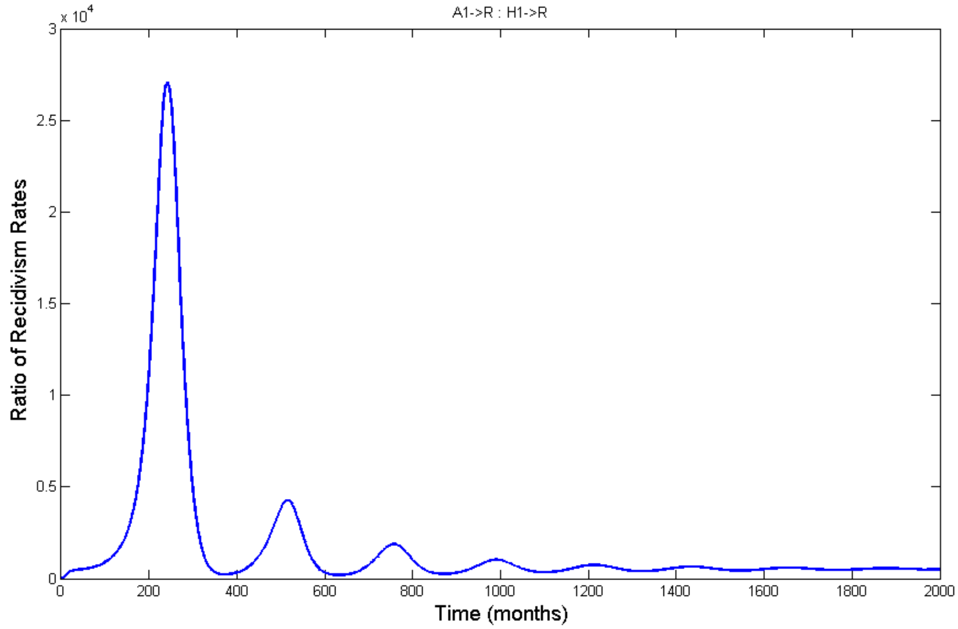

Figure 6: Ratio of Recidivism Rates $A_1 - > R : H_1 - > R$

Figure 6 shows the ratio of rates between the A_1 class and the H_1 class and the H_2 class and the H_1 class. $\frac{\alpha\theta_1 A_1}{\phi\theta_1 H_1} > 1$, meaning the rate at which the A_1 class goes into the recidivism class is much larger than the rate at which the H_1 class goes into the recidivism class. This Figure indicates completing both programs is significantly more important than only completing the program inside of prison.

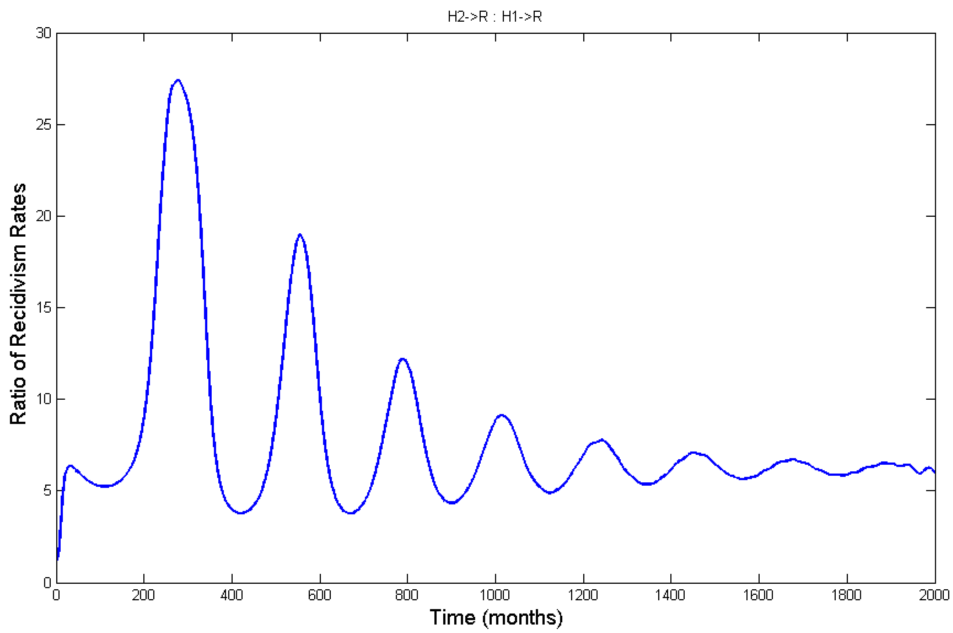

Figure 7: Ratio of Recidivism Rates $H_2->R : H_1->R$

Figure 7 shows the ratio of rates between the H_2 class and the H_1 class. $\frac{\phi\theta_2 H_2}{\phi\theta_1 H_1} > 1$, meaning the rate at which the H_2 class goes into the recidivism class is slightly larger than the rate at which the H_1 class goes into the recidivism class. This suggests how effective the Reentry Court program is compared to completing both programs.

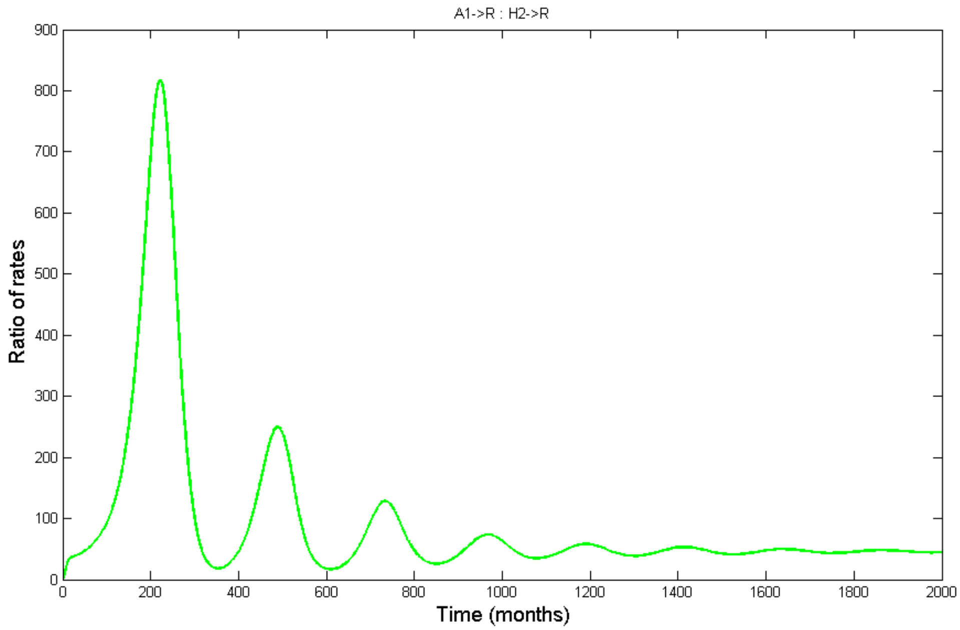

Figure 8: Ratio of Recidivism Rates $A_1 - > R : H_2 - > R$

Figure 8 shows the ratio of rates between the A_1 class and the H_2 class. $\frac{\alpha\theta_1 A_1}{\phi\theta_2 H_2} > 1$ suggests the rate at which the A_1 class goes into the recidivism class is much larger than the rate at which the H_2 class goes into the recidivism class. This implies that the Reentry Court program is more effective than the Basic Education program.

6 Conclusion

$\mathcal{R}_c$ is the reproductive number which is defined as the number of people one individual can influence during his time as a free criminal. We calculated $\mathcal{R}_c$ as being 1.98, which indicates every criminal can infect about two susceptibles. Our results suggests the number of people going back to prison decreases drastically for those individuals who complete the outside program in comparison to those who only complete the inside program, see Figure 4. However, this decrease in the recidivism class is at the expense of a slight increase in size of free criminals in the population. If the measure of effectiveness of a program is the size of the recidivism class, then the outside program out performs the inside prison program by a large number, as seen in Figure 4. However, if the measure of effectiveness of a program is the size of the criminal class, then the inside prison program slightly out performs the inside prison programs. Figure 10 and 11 of the simplified models give information on which program (educational, court reentry, or both) is more effective in reducing the rate of recidivism. In these figures we divide the ratios and if the plot is positive, then the class in the numerator has a higher recidivism rate. These results suggest that the outside program is much more effective than the inside program. In Figure 4, the steady state proportions of A_1, H_1, H_2, suggests that the H_1 class has the lowest proportion of recidivism and therefore completing both programs will result in the largest decrease of recidivism. Figure 11 suggests that those who only complete the Educational Program (A_1) will have a higher recidivism proportion when compared to H_2. From Figure 4 we can imply that the outside program (H_1 and H_2) has a greater effect on recidivism rates, this figure also suggests that the inside program has little effect on recidivism. Furthermore, the class of people who only do the inside program have a higher recidivism rate than both H_1 and A_1 but lower than A_2 which is the group of people who do not partake in any inside or outside program. We studied simplified models that can capture similar dynamics to the general model in Figure 1.

Further research can be done in comparing the cost of both the educational program and the reentry court program. Our analysis suggest that more money should be allocated into the H2 programs, however further cost analysis is suggested to support such claim.

7 Acknowledgments

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8 Appendix A

8.1 A_1 Analysis -R_0

To compute the basic reproductive number, we use the next generation operator. First, we find the $\mathcal{F}$ matrix based on new criminals,

$$\mathcal{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\beta CS}{N} + \frac{\beta_1 A_1 S}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we compute the F matrix,

$$F = \begin{bmatrix} \frac{\partial(x_1)}{\partial(C)} & \frac{\partial(x_1)}{\partial(I)} & \frac{\partial(x_1)}{\partial(A_1)} & \frac{\partial(x_1)}{\partial(R)} \\ \frac{\partial(x_2)}{\partial(C)} & \frac{\partial(x_2)}{\partial(I)} & \frac{\partial(x_2)}{\partial(A_1)} & \frac{\partial(x_2)}{\partial(R)} \\ \frac{\partial(x_3)}{\partial(C)} & \frac{\partial(x_3)}{\partial(I)} & \frac{\partial(x_3)}{\partial(A_1)} & \frac{\partial(x_3)}{\partial(R)} \\ \frac{\partial(x_4)}{\partial(C)} & \frac{\partial(x_4)}{\partial(I)} & \frac{\partial(x_4)}{\partial(A_1)} & \frac{\partial(x_4)}{\partial(R)} \end{bmatrix} = \begin{bmatrix} \frac{\beta S}{N} & 0 & \frac{\beta_1 S}{N} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The $\mathcal{V}$ matrix tracks the inflow and outflow of criminals from each compartment

$$\mathcal{V} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} C\sigma + C\mu \\ I\mu + I\gamma - C\sigma \\ A_1\theta_1 + A_1\mu - Ip\gamma \\ R\mu + R\gamma - A_1\alpha\theta_1 \end{bmatrix}$$

Then we compute the V matrix,

$$V = \begin{bmatrix} \frac{\partial(y_1)}{\partial(C)} & \frac{\partial(y_1)}{\partial(I)} & \frac{\partial(y_1)}{\partial(A_1)} & \frac{\partial(y_1)}{\partial(R)} \\ \frac{\partial(y_2)}{\partial(C)} & \frac{\partial(y_2)}{\partial(I)} & \frac{\partial(y_2)}{\partial(A_1)} & \frac{\partial(y_2)}{\partial(R)} \\ \frac{\partial(y_3)}{\partial(C)} & \frac{\partial(y_3)}{\partial(I)} & \frac{\partial(y_3)}{\partial(A_1)} & \frac{\partial(y_3)}{\partial(R)} \\ \frac{\partial(y_4)}{\partial(C)} & \frac{\partial(y_4)}{\partial(I)} & \frac{\partial(y_4)}{\partial(A_1)} & \frac{\partial(y_4)}{\partial(R)} \end{bmatrix} = \begin{bmatrix} \sigma + \mu & 0 & 0 & 0 \\ -\sigma & \gamma + \mu & 0 & 0 \\ 0 & -p\gamma & \mu + \theta_1 & 0 \\ 0 & 0 & -\alpha\theta_1 & \gamma + \mu \end{bmatrix}$$

Then we compute V^{-1},

$$V^{-1} = \begin{bmatrix} \frac{1}{\sigma+\mu} & 0 & 0 & 0 \\ \frac{\sigma}{(\gamma+\mu)(\sigma+\mu)} & \frac{1}{(\gamma+\mu)} & 0 & 0 \\ \frac{p\sigma\gamma}{(\gamma+\mu)(\sigma+\mu)(\theta_1+\mu)} & \frac{p\gamma}{(\gamma+\mu)(\theta_1+\mu)} & \frac{1}{\theta_1+\mu} & 0 \\ \frac{p\sigma\gamma\alpha\theta_1}{(\gamma+\mu)^2(\sigma+\mu)(\theta_1+\mu)} & \frac{p\sigma\gamma\alpha\theta_1}{(\gamma+\mu)^2(\theta_1+\mu)} & \frac{\alpha\theta_1}{(\gamma+\mu)(\theta_1+\mu)} & \frac{1}{(\gamma+\mu)} \end{bmatrix}$$

To find R_0 we evaluate both F and V^{-1} at our crime-free equilibrium and find the largest eigenvalue along the diagonal in the product of the two matrices.

$$\begin{aligned}
R_0 &= \rho(FV^{-1}) \\
&= \rho\left(\begin{bmatrix} \beta & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sigma+\mu} & 0 & 0 & 0 \\ \frac{\sigma}{(\gamma+\mu)(\sigma+\mu)} & \frac{1}{(\gamma+\mu)} & 0 & 0 \\ \frac{p\sigma\gamma}{(\gamma+\mu)(\sigma+\mu)(\theta_1+\mu)} & \frac{p\gamma}{(\gamma+\mu)(\theta_1+\mu)} & \frac{1}{\theta_1+\mu} & 0 \\ \frac{p\sigma\gamma\alpha\theta_1}{(\gamma+\mu)^2(\sigma+\mu)(\theta_1+\mu)} & \frac{p\sigma\gamma\alpha\theta_1}{(\gamma+\mu)^2(\theta_1+\mu)} & \frac{\alpha\theta_1}{(\gamma+\mu)(\theta_1+\mu)} & \frac{1}{(\gamma+\mu)} \end{bmatrix}\right) \\
&= \rho\left(\begin{bmatrix} \frac{\beta}{\sigma+\mu} + \frac{\beta_1 p\sigma\gamma}{(\gamma+\mu)(\sigma+\mu)(\theta_1+\mu)} & \frac{\beta_1 p\gamma}{(\gamma+\mu)(\theta_1+\mu)} & \frac{\beta_1}{\theta_1+\mu} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&= \frac{\beta}{\sigma+\mu} + \frac{\beta_1 p\sigma\gamma}{(\gamma+\mu)(\sigma+\mu)(\theta_1+\mu)}
\end{aligned}$$

8.1.1 Finding Equilibria

By setting all ODE's equal to zero we find our disease-free equilibrium to be

$$(N, 0, 0, 0, 0)$$

To find our endemic equilibria by setting all of the following equations in terms of C.

$$\begin{aligned}
S' &= \Lambda - \frac{\beta CS}{N} - \frac{\beta_1 A_1 S}{N} - \mu S \\
C' &= \frac{\beta CS}{N} + \frac{\beta_1 A_1 S}{N} - \mu C - \sigma C \\
I' &= \sigma C - \gamma I - \mu I \\
A_1' &= \gamma p I - \theta_1 A_1 - \mu A_1 \\
R' &= \theta_1 \alpha A_1 - \gamma R - \mu R
\end{aligned}$$

In order to solve for S we add S'+ C' and set it equal to zero.

$$\begin{aligned}
0 &= \Lambda - \frac{\beta C^* S}{N} - \frac{\beta_1 A_1 S}{N} - \mu S + \frac{\beta C^* S}{N} + \frac{\beta_1 A_1 S}{N} - \mu C^* - \sigma C^* \\
0 &= \Lambda - \mu S - \mu C^* - \sigma C^*
\end{aligned}$$

We then solve for the variable S.

$$\begin{aligned}\mu S &= \Lambda - \mu C^* - \sigma C^* \\ S &= \frac{\Lambda}{\mu} - C^* - \frac{\sigma C^*}{\mu} \\ &= \frac{1}{\mu}(\lambda - (\mu + \sigma)C^*)\end{aligned}$$

Next we solve for I in terms of C^*.

$$\begin{aligned}0 &= \sigma C^* - I\mu I\gamma \\ 0 &= \sigma C^* + I(-\mu - \gamma) \\ I &= \frac{\sigma C^*}{s_2}\end{aligned}$$

Now we will be putting A_1 in terms of C^*.

$$\begin{aligned}0 &= -\gamma p I - \mu A_1 - \alpha\theta_1 A_1 - (1-\alpha)\theta_1 A_1 \\ 0 &= \gamma - \mu A_1 - \alpha\theta_1 A_1 - \theta_1 A_1 + \alpha\theta_1 A_1 \\ 0 &= \gamma p I - \mu A_1 - \theta_1 A_1\end{aligned}$$

We simultaneously substitute our I term from above as well as solve for A_1:

$$A_1 = \frac{\gamma p \sigma C^*}{s_2 s_1}$$

We have to put our R equation in terms of C^* as well.

$$\begin{aligned}0 &= \alpha\theta_1 A_1 - \gamma R - \mu R \\ 0 &= \alpha\theta_1 A_1 - R(\gamma + \mu) \\ R &= \frac{\alpha\theta_1 A_1}{\gamma + \mu}\end{aligned}$$

We then go on to substitute A_1 with the solution above in order to put R in terms of C^*.

$$R = \frac{\alpha\theta_1(\gamma p \sigma C^*)}{s_2 s_1^2}$$

Now that we have put all of our equations in terms of C^*. We will be taking our S' equation and replacing the values of A_1 and S so that all our equation is in terms of C^*.

$$S' = \Lambda - \frac{\beta C^*}{N}\left(\frac{\Lambda}{\mu} - C^* - \frac{\sigma C^*}{\mu}\right) - \beta_1 \frac{\gamma p \sigma C^*}{s_2 s_3}\left(\frac{\Lambda}{\mu} - C^* - \frac{\sigma C^*}{\mu}\right) - \mu\left(\frac{\Lambda}{\mu} - C^* - \frac{\sigma C^*}{\mu}\right)$$

In order to simplify the equation we will let:

$$\begin{aligned} s_1 &= \mu + \theta_1 \\ s_2 &= \mu + \gamma \\ s_3 &= \mu + \sigma \end{aligned}$$

We now substitute our $s_1 s_2 s_3$ into our equation below in order to simplify.

$$\begin{aligned} 0 &= \Lambda - \frac{\beta C^*}{N}\left(\frac{\Lambda}{\mu} - \frac{C^*}{\mu}s_3\right) - \beta_1 \frac{\gamma p \sigma C^*}{s_2 s_1}\left(\frac{\Lambda}{\mu} - \frac{C^*}{\mu}s_3\right) - \mu\left(\frac{\Lambda}{\mu} - \frac{C^*}{\mu}s_3\right) \\ 0 &= \Lambda - \beta C^* + \frac{\beta C^{*2}}{N\mu}s_3 - \beta_1 \frac{\gamma p \sigma C^*}{s_2 s_1}\left(\frac{\Lambda}{\mu} - \frac{C^*}{\mu}s_3\right) - \Lambda - C^* s_3 \end{aligned}$$

Using maple we set the equation to zero and solve for C^*. We get a a value for C^* giving us the endemic equilibrium:

$$C^* = N\frac{(\beta\mu s_3 s_1 + \beta_3 p_2 \sigma\gamma\Lambda - \mu s_3^2 s_1)}{s_3(\beta s_3 s_1 + \beta_3 p_2 \sigma \nu N)}$$

8.1.2 Stability of Disease-Free Equilibrium

$$\begin{aligned} S' &= \Lambda - \frac{\beta C S}{N} - \frac{\beta 1 A_1 S}{N} - \mu S \\ C' &= \frac{\beta C S}{N} + \frac{\beta 1 A_1 S}{N} - C\sigma - C\mu \\ I' &= C\sigma - I\mu - Ip\gamma - I(1-p)\gamma \\ A_1' &= Ip\gamma - A_1\mu - A_1\theta_1 \\ R' &= A_1\alpha\theta_1 - R\gamma - R\mu \end{aligned}$$

We then find the Jacobian of the system of differential equations and evaluate it at our crime-free equilibrium.

$$\begin{aligned}
J(S^*, C^*, I^*, A_1^*, R^*) &= \begin{bmatrix} \frac{\partial(S')}{\partial(S)} & \frac{\partial(S')}{\partial(C)} & \frac{\partial(S')}{\partial(I)} & \frac{\partial(S')}{\partial(A_1)} & \frac{\partial(S')}{\partial(R)} \\ \frac{\partial(C')}{\partial(S)} & \frac{\partial(C')}{\partial(C)} & \frac{\partial(C')}{\partial(I)} & \frac{\partial(C')}{\partial(A_1)} & \frac{\partial(C')}{\partial(R)} \\ \frac{\partial(I')}{\partial(S)} & \frac{\partial(I')}{\partial(C)} & \frac{\partial(I')}{\partial(I)} & \frac{\partial(I')}{\partial(A_1)} & \frac{\partial(I')}{\partial(R)} \\ \frac{\partial(A_1')}{\partial(S)} & \frac{\partial(A_1')}{\partial(C)} & \frac{\partial(A_1')}{\partial(I)} & \frac{\partial(A_1')}{\partial(A_1)} & \frac{\partial(A_1')}{\partial(R)} \\ \frac{\partial(R')}{\partial(S)} & \frac{\partial(R')}{\partial(C)} & \frac{\partial(R')}{\partial(I)} & \frac{\partial(R')}{\partial(A_1)} & \frac{\partial(R')}{\partial(R)} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\beta C}{N} - \frac{\beta_1 A_1}{N} - \mu & -\frac{\beta S}{N} & 0 & -\frac{\beta_1 S}{N} & 0 \\ \frac{\beta C}{N} + \frac{\beta_1 A_1}{N} & \frac{\beta S}{N} - \sigma - \mu & 0 & \frac{\beta_1 S}{N} & 0 \\ 0 & \sigma & -\mu - p\gamma - (1-p)\gamma & 0 & 0 \\ 0 & 0 & p\gamma & -\mu - \theta_1 & 0 \\ 0 & 0 & 0 & \alpha\theta_1 & -\gamma - \mu \end{bmatrix}
\end{aligned}$$

$$J(N,0,0,0,0) = \begin{bmatrix} -\mu & -\beta & 0 & -\beta_1 & 0 \\ 0 & \beta - \sigma - \mu & 0 & \beta_1 & 0 \\ 0 & \sigma & -\mu - p\gamma - (1-p)\gamma & 0 & 0 \\ 0 & 0 & p\gamma & -\mu - \theta_1 & 0 \\ 0 & 0 & 0 & \alpha\theta_1 & -\gamma - \mu \end{bmatrix}$$

Since the first and last columns have only one nonzero term, we can look at the $3x3$ matrix inside of the Jacobian matrix to determine our characteristic polynomial.

$$\begin{pmatrix} \beta - \sigma - \mu & 0 & \beta_1 \\ \sigma & -\mu - p\gamma - (1-p)\gamma & 0 \\ 0 & p\gamma & -\mu - \theta_1 \end{pmatrix}$$

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

where,

$$\begin{aligned}
a_1 &= 3\mu + \theta_1 + \gamma - \beta + \sigma \\
a_2 &= 2\mu\gamma + 3\mu^2 - 2\mu\beta + 2\mu\sigma + \theta_1\gamma + 2\theta_1\mu - \theta_1\beta + \theta_1\sigma - \gamma\beta + \gamma\sigma \\
a_3 &= -p\gamma\sigma\beta 1 - \mu\gamma\beta + \mu\gamma\sigma + \mu^2\gamma - \mu^2\beta + \mu^2\sigma + \mu^3 - \theta_1\gamma\beta + \theta_1\gamma\sigma + \theta_1\gamma\mu - \theta_1\mu\beta + \theta_1\mu\sigma + \theta_1\mu^2
\end{aligned}$$

Let,

$$\begin{aligned}
S_1 &= (\mu + \sigma) \\
S_2 &= (\mu + \gamma) \\
S_3 &= (\mu + \theta_1)
\end{aligned}$$

This gives us,

$$\begin{aligned}
a_1 &= S_1 + S_2 + S_3 - \beta \\
a_2 &= (S_2 + S_3)(S_1 - \beta) + S_2 S_3 \\
a_3 &= S_1 S_2 S_3 - \beta S_2 S_3 - \beta_1 p \sigma \gamma
\end{aligned}$$

To determine the stability of the crime-free equilibrium, we use the Routh-Hurwitz Criterion.

Theorem 2. Routh-Hurwitz Criterion *If the a_i coefficients of the third degree polynomial $x^3 + a_1 x^2 + a_2 x + a_3$ satisfy the following,*

1. $a_1 > 0$
2. $a_3 > 0$
3. $a_1 a_2 > a_3$, *where* $a_2 > 0$

then our crime-free equilibrium is locally asymptotically stable.

First we show that $a_1 > 0$,

$$\begin{aligned}
a_1 = S_1 + S_2 + S_3 - \beta &> 0 \\
S_1 + S_2 + S_3 &> \beta \\
1 + \frac{S_2}{S_1} + \frac{S_3}{S_1} &> \frac{\beta}{S_1}
\end{aligned}$$

Since $R_0 = \frac{\beta}{S_1} + \frac{\beta_1 p \sigma \gamma}{S_1 S_2 S_3}$,

$$\frac{\beta}{S_1} < R_0 < 1 < 1 + \frac{S_2}{S_1} + \frac{S_3}{S_1}$$

Therefore, $\frac{\beta}{S_1} < 1 + \frac{S_2}{S_1} + \frac{S_3}{S_1}$. Now we show $a_3 > 0$,

$$\begin{aligned}
R_0 = \frac{\beta}{S_1} + \frac{\beta_1 p \sigma \gamma}{S_1 S_2 S_3} &< 1 \\
\beta S_2 S_3 + \beta_1 p \sigma \gamma &< S_1 S_2 S_3 \\
0 &< S_1 S_2 S_3 - \beta S_2 S_3 - \beta_1 p \sigma \gamma
\end{aligned}$$

We have that $a_3 = S_1 S_2 S_3 - \beta S_2 S_3 - \beta_1 p \sigma \gamma$. Thus $a_3 > 0$.

Lastly we show that $a_1a_2 > a_3$, since $R_0 = \frac{\beta}{S_1} + \frac{\beta_1 p\sigma\gamma}{S_1S_2S_3}$ and $S_1, S_2, S_3 > 0$,

$$\begin{aligned}
\frac{\beta}{S_1} &< R_0 \\
-\frac{\beta}{S_1} &> -R_0 \\
1-\frac{\beta}{S_1} &> 1-R_0 \\
(S_1S_2S_3)(1-\frac{\beta}{S_1}) &> (S_1S_2S_3)(1-R_0) \\
(S_1S_2S_3)(1-\frac{\beta}{S_1}) &> a_3
\end{aligned}$$

So it suffices to prove that $a_1a_2 > (S_1S_2S_3)(1-\frac{\beta}{S_1})$.

$$\begin{aligned}
a_1a_2 &> (S_1S_2S_3)(1-\frac{\beta}{S_1}) \\
\frac{a_1a_2}{(S_1S_2S_3)(1-\frac{\beta}{S_1})} &> 1 \\
\frac{a_2}{(S_1S_2S_3)(1-\frac{\beta}{S_1})} &> \frac{1}{a_1} \\
\frac{(S_2+S_3)(S_1-\beta)+S_2S_3}{(S_1S_2S_3)(1-\frac{\beta}{S_1})} &> \frac{1}{S_1+S_2+S_3-\beta} \\
\frac{S_2S_3}{(S_1S_2S_3)(1-\frac{\beta}{S_1})} + \frac{(S_2+S_3)(S_1-\beta)}{(S_2S_3)(S_1-\beta)} &> \frac{1}{S_1+S_2+S_3-\beta} \\
\frac{1}{S_1-\beta} + \frac{S_2+S_3}{S_2S_3} &> \frac{1}{S_1+S_2+S_3-\beta} \\
\frac{1}{S_1-\beta} + \frac{1}{S_2} + \frac{1}{S_3} &> \frac{1}{S_1+S_2+S_3-\beta} \\
(S_1+S_2+S_3-\beta)(\frac{1}{S_1-\beta} + \frac{1}{S_2} + \frac{1}{S_3}) &> 1 \\
\frac{S_2+S_3}{S_1-\beta} + 1 + \frac{S_1-\beta+S_3}{S_2} + 1 + \frac{S_1-\beta+S_2}{S_3} + 1 &> 1 \\
\frac{S_2+S_3}{S_1-\beta} + \frac{S_1-\beta+S_3}{S_2} + \frac{S_1-\beta+S_2}{S_3} + 3 &> 1
\end{aligned}$$

Since $\frac{\beta}{S_1} < 1 \Rightarrow \beta < S_1 \Rightarrow 0 < S_1 - \beta$, this implies that $\frac{S_2+S_3}{S_1-\beta} + \frac{S_1-\beta+S_3}{S_2} + \frac{S_1-\beta+S_2}{S_3} > 0$. Thus $a_1a_2 > a_3$ and the CFE is asymptotically stable.

9 Appendix B

9.1 H_1 vs H_2 Analysis - Basic Reproductive Number

In this section, we compute the $\mathcal{R}_0$ for the model which compares the populations H_1 and H_2. The partial derivative of the matrix $\mathcal{F}$ evaluated at the disease free equilibrium: Calculating $\mathcal{R}_0$:

$$\mathcal{F} = \begin{bmatrix} \frac{\beta CS}{N} + \frac{\beta_3 H_2 S}{N} + \frac{\beta_4 H_1 S}{N} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad J_1|_{(N,0,0,0,0,0)} = \begin{bmatrix} \beta & \beta_3 & 0 & \beta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The $\mathcal{V}$ matrix represents all the variables left in the infected classes after getting your $\mathcal{F}$ matrix. The partial derivative of $\mathcal{V}$ evaluated at the disease free equilibrium.

$$\mathcal{V} = \begin{bmatrix} \sigma C + \mu C \\ -\sigma C + \mu I + \gamma I \\ -(1-r_1)\gamma I + \mu H_1 + \theta_1 H_1 \\ -\gamma r_1 I + \mu H_2 + \theta_1 H_2 \\ -\phi\theta_1 H_1 - \phi\theta_1 H_2 + \gamma R + \mu R \end{bmatrix},$$

$$J_2|_{(N,0,0,0,0,0)} = \begin{bmatrix} \sigma + \mu & 0 & 0 & 0 & 0 \\ -\sigma & 0 & \mu + \gamma(1-r_1) + \gamma r_1 & 0 & 0 \\ 0 & 0 & -(1-r_1)\gamma & \phi\theta_1 + \mu + \theta_1(1-\phi) & 0 \\ 0 & \phi\theta_1 + \mu + \theta_1(1-\phi) & -\gamma r_1 & 0 & 0 \\ 0 & -\phi\theta_1 & 0 & -\phi\theta_1 & \gamma + \mu \end{bmatrix}.$$

In order to compute $\mathcal{R}_0$ first we need to take the inverse of J_2. We take the partial derivative of J_1 and $(J_2)^{-1}$ and multiply them. The following matrix is what the product:

$$\mathcal{R}_0 = J_1(J_2)^{-1} = (\begin{bmatrix} \frac{\beta}{\sigma+\mu} + \frac{\beta_3 r_1 \gamma\sigma}{\mu+\theta_1)(\sigma+\mu)(\gamma+\mu)} & \frac{\beta_3\gamma r_1}{(\gamma+\mu)(\mu+\theta_1)} & 0 & \frac{\beta_3}{\mu+\theta_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{R}_0 = \frac{\beta}{\sigma+\mu} + \frac{\beta_3 r_1 \gamma\sigma}{(\mu+\theta_1)(\mu+\sigma)(\mu+\gamma)}$$

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