Political Recruitment via Television Ads in a Two-Party System

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Abstract

The dynamics of recruitment within a two political party system that influence individuals through party member contact and television advertising are explored via a mathematical contagion model that includes a population of committed party members and a population of "undecided" or "susceptible" individuals. The contagion model supports multiple outcomes including two-party coexistence. In order to examine the effect of ads and membership on the population in a two party system, we first examine each influence separately. This model supports ads having a signification impact on the amount of members in each party, addresses the impact of resources and sets the stage for framework expansions for other recruitment strategies.

1 Introduction

In a two party system, like in the United States, parties use advertising to win elections. Candidates running for representative, senator, or president will sometimes put large amounts of money into television advertisements. In the 2008 Presidential election, Barack Obama spent $280 million on television advertising from Jan 2007 to Nov 2008, while John McCain spent $134 million [1] [2]. This money was used to try and entice as many party members to their side. Understanding the connection between the amount of voters and the amount of ads aired will shine light on how each party effects the opposing one when competing for members.

Competitive two party dynamics are complex [10] [22] [11] [17]. Here, we use a trivial example that focuses on the study of (primarily) the role of television advertisement on leaning individuals and in the recruitment of new voters. This work is motivated and preliminary of what could be done when the complexities associated with political dynamics is explored gradually.

In this paper we use a deterministic mathematical model to study how television ads change the amount of members in two parties. The model can be seen as an contagion model focusing on two diseases. Party members represent the infected population while potential members are susceptible. Advertisements target the susceptible classes. This model assumes that people can only join one of two parties, and that susceptible individuals may only be influenced by television ads or via contacts with party members.

The complexity of political dynamics is difficult to model. We make some assumptions in order to simplify the interaction between voters, party members and television ads. We assume there are two types of voters. There are those voters who favor one party over the other. These people have an existing preference and are easily persuaded by their preferred party. The second type of voters are those that are open to both parties. They have equal likelihood of joining either party. The television ads try to persuade as many of these undecided individuals as possible.

Members can leave a party and go to one of two susceptible classes . Some will leave the party but still have a preference to that party. Others will leave and be open-minded to both parties. Ads are used to persuade individuals to make them more likely to join one of the two parties. If one party is running more effective ads than another it will persuade a larger proportion of the middle population.

In the case of politics, those who have a preference towards one party will not listen to the opposing party's members [15, 6, 7]. Therefore there is no connection between the two in the model. Only those who have no preference to either party will be able to communicate to members from both parties. The incorporation of births and deaths add an interesting aspect to the model. Parents have a significant impact on which political party their offspring choose [9]. Births from party members have a preference towards the same party but are not members themselves. Those born will have the same preference as their parents. In our model we choose to look at the proportion of the population that prefer each party.

Partisan identification is the most important aspect when individuals vote [10]. We can assume the population's identification looking at the general population's identification. Candidate image and issue positions change from person to person [16] [3]. We assume that each party has similar reputation levels. Potential members are treated as isolated individuals. They are not influenced by groups such as religions, economics or social background.

Our model includes the idea of positive and negative ads. We assume negative ads target those who prefer a party, and positive ads target those in the middle class. Positive ads have a higher influence on susceptible individual's decision while negative ads have lower influence and can even alienate voters from either party [8]. We assume ads targeting the preferred classes are less effective than those targeting the middle class.

This model was based on a model that looked at the spread of third party support [21] and a competition model for beer companies with television ads[14]. Advertising for products is similar to advertising for a candidate [25]. Both use psychological methods to invoke emotions, causing viewers to act. We use the mechanism for ads from the one developed by Chacon et al [14].

In order to answer the overall effectiveness of ads and member contacts we must examine the effects of each on their own. First we look at a situation where neither party is running ads. New members can be recruited by existing ones but there will be no shift in preference in the susceptible population. The second situation looks at a single party with influence from members and television ads.

2 Compartmental Model

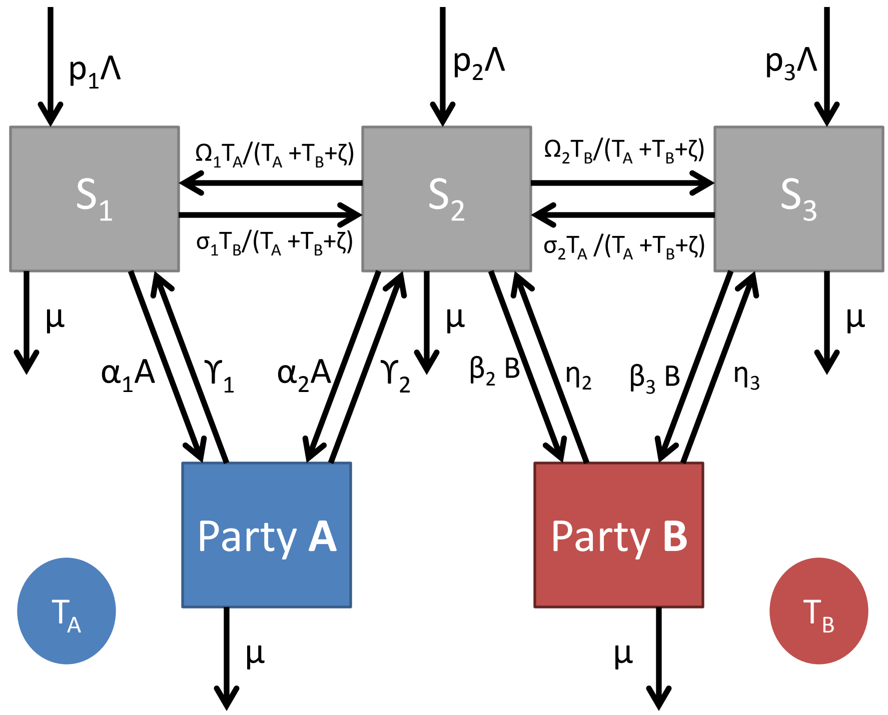
Political dynamics are complex so we use a caricature with simplification. We are interested in how both television ads and contact from party members effect each party. This model was based off of two papers, One paper by Charcon et al that focuses on the advertising mechanism [14]. The second by Romero et al focuses on party member recruitment through contacts [21]. We add susceptible individuals with preferences.

The situation can be described as a two strain SIS model with various levels of susceptibility. Members can "infect" individuals, causing them to join their party. Individuals are more susceptible if they have seen many ads from a party but these are being combated by ads from the opposing party. Individuals can leave parties by "recovering" or leave the system entirely. Those that leave to one of the extreme classes will have a preference to their party. Those that leave to the middle class do not have a preference and are open to both parties.

Those that are in the extreme populations are easier to pursued then those in the middle class. The infection rate for those in the preferred class is higher than the middle one. Similarly it is more likely that someone will exit a party through the middle population rather than though one of the preferred classes.

The amount of ads each party runs is based on the proportion of members each party has. If party A has larger amount of member compared to party B, they will air less ads. There is a limited number of ads each person can be exposed to throughout the day. So the effectiveness of the ads on individuals is proportional to the amount of total ads aired.

The following is the general form of the model:



$$
\begin{aligned}
\frac{dS_1}{dt} =& p_1\Lambda + \gamma_1 A + \Omega_1 \left(\frac{T_A}{T_A + T_B + \zeta} \right) S_2 - \alpha_1 A S_1 - \sigma_1 \left(\frac{T_B}{T_A + T_B + \zeta} \right) S_1 - \mu S_1 \\
\frac{dS_2}{dt} =& p_2\Lambda + \gamma_2 A + \eta_2 B + \sigma_1 \left(\frac{T_B}{T_A + T_B + \zeta} \right) S_1 + \sigma_2 \left(\frac{T_A}{T_A + T_B + \zeta} \right) S_3 \\
& - \alpha_2 A S_2 - \beta_2 B S_2 - \Omega_1 \left(\frac{T_A}{T_A + T_B + \zeta} \right) S_2 - \Omega_2 \left(\frac{T_B}{T_A + T_B + \zeta} \right) S_2 - \mu S_2 \\
\frac{dS_3}{dt} =& p_3\Lambda + \eta_3 B + \Omega_2 \left(\frac{T_B}{T_A + T_B + \zeta} \right) S_2 - \beta_3 B S_3 - \sigma_2 \left(\frac{T_A}{T_A + T_B + \zeta} \right) S_3 - \mu S_3 \\
\frac{dA}{dt} =& A(\alpha_2 S_2 + \alpha_1 S_1 - (\mu + \gamma_1 + \gamma_2)) \\
\frac{dB}{dt} =& B(\beta_3 S_3 + \beta_2 S_2 - (\mu + \eta_1 + \eta_2)) \\
\frac{dT_A}{dt} =& r_1 \left(1 - \frac{A}{A + B + \omega} \right) \left(1 - \frac{T_A}{K_1} \right) T_A \\
\frac{dT_B}{dt} =& r_2 \left(1 - \frac{B}{A + B + \omega} \right) \left(1 - \frac{T_B}{K_2} \right) T_B
\end{aligned}
$$

Class	Definition	Case Study	Reference
S_1	Susceptible population that prefers party A	222,800,000(.35)	[12] [20]
S_2	Susceptible population that prefers neither party	222,800,000(.34)	[12] [20]
S_3	Susceptible population that prefers party B	222,800,000(.28)	[12] [20]
A	Members for party A	700,000	[24]
B	Members for party B	2,000,000	[4]
T_A	Number of ads from party A	1	
T_B	Number of ads from party B	1	
Parameter	Definition	Estimate	Reference
α_1	Rate that individuals from S_1 join party A	$(4.43 * 10^{-9})(2/3)$	[12] [24]
α_2	Rate that individuals from S_2 join party A	$(4.43 * 10^{-9})(1/3)$	[12] [24]
β_2	Rate that individuals from S_2 join party B	$(4.43 * 10^{-9})(1/3)$	[12] [4]
β_3	Rate that individuals from S_3 join party B	$(4.43 * 10^{-9})(2/3)$	[12] [4]
γ_1	Rate that individuals leave party A to S_1	$3.21 * 10^{-10}$	
γ_2	Rate that individuals leave party A to S_2	$6.41 * 10^{-10}$	
η_2	Rate that individuals leave party B to S_2	$6.41 * 10^{-10}$	
η_3	Rate that individuals leave party B to S_3	$3.21 * 10^{-10}$	
μ	death rate	$1/(60 * 365)$	[19]
Λ	recruitment rate	11637	[28]
p_1	proportion of recruitment into S_1	.35	[20]
p_2	proportion of recruitment into S_2	.34	[20]
p_3	proportion of recruitment into S_3	.28	[20]
σ_1	Effectiveness of ads on S_1 from McCain	.0128	[5] [23]
σ_2	Effectiveness of ads on S_3 from party A	.0289	[5] [23]
Ω_1	Effectiveness of ads on S_2 from party A	.1315	[5] [23]
Ω_2	Effectiveness of ads on S_2 from party B	.0627	[5] [23]
K_1	Maximum number of ads party A can afford at time t	1904/673	[23]
K_2	Maximum number of ads party B can afford at time t	911/673	[23]
ζ	numerical value between 0 and 1 for a non-zero denominator	1	
ω	numerical value between 0 and 1 for a non-zero denominator	1	

2.1 Model With No Advertising

The model above is complex as a whole. Trying to examine both influences at once is difficult. So we first look at just influence by party members. This model assumes that both T_A and T_B are zero. The system now take this form:

$$\begin{cases} \frac{dS_1}{dt} = \Lambda p_1 + \gamma_1 A - \mu S_1 - \alpha_1 A S_1 \\ \frac{dS_2}{dt} = \Lambda p_2 - \mu S_2 + \gamma_2 A + \eta_2 B - \alpha_2 A S_2 - \beta_2 B S_2 \\ \frac{dS_3}{dt} = \Lambda p_3 + \eta_3 B - \mu S_3 - \beta_3 B S_3 \\ \frac{dA}{dt} = A(\alpha_2 S_2 + \alpha_1 S_1 - (\mu + \gamma_1 + \gamma_2)) \\ \frac{dB}{dt} = B(\beta_3 S_3 + \beta_2 S_2 - (\mu + \eta_2 + \eta_3)) \end{cases} \tag{1}$$

People join party A and B through contact with current members, by social interaction. New recruits do not switch susceptible classes unless it is through joining a party and then leaving. First we want to show this model (1) is well-posed, i.e, the trajectories remain positive and bounded. We have the following lemma.

Lemma 1. *The region defined by*

$$\Omega = \left\{ (S_1, S_2, S_3, A, B) \in \mathbb{R}^5_+ \middle| S_1 + S_2 + S_3 + A + B \leq \frac{\Lambda}{\mu} \right\}$$

is a compact positively invariant set for the system (1). Moreover, the set $\{(S_1, S_2, S_3, 0, 0) \in \mathbb{R}^5 +\}$ *is a stable manifold for the system (1).*

Proof. Notice that

$$\begin{aligned} \tfrac{dS_1}{dt}\Big|_{S_1=0} &= \Lambda p_1 + \gamma_1 A > 0 \\ \tfrac{dS_2}{dt}\Big|_{S_2=0} &= \Lambda p_2 + \gamma_2 A + \eta_2 B \geq 0 \\ \tfrac{dS_3}{dt}\Big|_{S_3=0} &= \Lambda p_3 + \eta_3 B > 0 \\ \tfrac{dA}{dt}\Big|_{A=0} &= 0 \\ \tfrac{dB}{dt}\Big|_{B=0} &= 0 \end{aligned}$$

This implies that the set $\{S_1 \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0, A \geq 0, B \geq 0\}$ is positively invariant for the system (1). Furthermore, we have, the dynamics of the total population is:

$$\frac{dN(t)}{dt} = \Lambda - \mu N$$

This implies that $\limsup_{t \to +\infty} N(t) = \dfrac{\Lambda}{\mu}$. This means, at any time t, $S_1(t) + S_2(t) + S_3(t) + A(t) + B(t) \leq \frac{\Lambda}{\mu}$. Therefore, the trajectories are bounded. The set Ω is clear a compact. $\square$

Now we examine equilibrium points. The Model 1 has fixed equilibrium points that do not depend on the initial state of the system. We used mathematica to solve our equations. There are four equilibrium points,

$$E_0 = \left(\frac{\Lambda p_1}{\mu}, \frac{\Lambda p_2}{\mu}, \frac{\Lambda p_3}{\mu}, 0, 0 \right),$$

$$E_1 = \left(\frac{(\mu + \gamma_1 + \gamma_2) - \bar{S_2}\alpha_2}{\alpha_1}, \bar{S_2}, \frac{\Lambda p_3}{\mu}, \frac{\mu \bar{S_2} - \Lambda p_2}{\alpha_2 \bar{S_2} + \gamma_2}, 0 \right),$$

$$E_2 = \left(\frac{\Lambda p_1}{\mu}, S_2^{\#}, \frac{(\mu + \eta_2 + \eta_3) - S_2^{\#}\beta_2}{\beta_3}, 0, \frac{\mu S_2^{\#} - \Lambda p_2}{\beta_2 S_2^{\#} + \eta_2} \right),$$

$$E_3 = \left(\frac{(\mu + \gamma_1 + \gamma_2) - S_2^*\alpha_2}{\alpha_1}, S_2^*, \frac{(\mu + \eta_2 + \eta_3) - S_2^*\beta_2}{\beta_3}, \frac{S_2^*\mu\alpha_2 + \Lambda p_1 \alpha_1 - \mu(\mu + \gamma_1 + \gamma_2)}{\alpha_1(\mu - S_2^*\alpha_2 + \gamma_2)}, \frac{S_2^*\mu\beta_2 + \Lambda p_3 \beta_3 - \mu(\mu + \eta_2 + \eta_3)}{\beta_3(\mu - S_2^*\beta_2 + \eta_2)} \right)$$

Where $\bar{S_2}$ and $S_2^{\#}$ are,

$$\begin{aligned} \bar{S_2} =& \frac{\alpha_1(\Lambda p_1 \alpha_2 + \Lambda p_2 \alpha_2 + \mu(\mu + \gamma_2)) - \mu\alpha_2(\mu + \gamma_1 + 2\gamma_2)}{2\mu(\alpha_1 - \alpha_2)\alpha_2} \\ &\pm \frac{\sqrt{(\alpha_1(\Lambda p_1 \alpha_2 + \Lambda p_2 \alpha_2 + \mu(\mu + \gamma_2)) - \mu\alpha_2(\mu + \gamma_1 + 2\gamma_2))^2 + 4\mu(\alpha_1 - \alpha_2)\alpha_2(\gamma_2(\Lambda p_1 \alpha_1 - \mu(\mu + \gamma_1 + \gamma_2)) + \Lambda p_2 \alpha_1(\mu + \gamma_2))}}{2\mu(\alpha_1 - \alpha_2)\alpha_2} \\ S_2^{\#} =& \frac{\beta_3(\Lambda p_3 \beta_2 + \Lambda p_2 \beta_2 + \mu(\mu + \eta_2)) - \mu\beta_2(\mu + \eta_3 + 2\eta_2)}{2\mu(\beta_2 - \beta_3)\beta_2} \\ &\pm \frac{\sqrt{(\beta_3(\Lambda p_3 \beta_2 + \Lambda p_2 \beta_2 + \mu(\mu + \eta_2)) - \mu\beta_2(\mu + \eta_3 + 2\eta_2))^2 + 4\mu(\beta_2 - \beta_3)\beta_2(\eta_2(\Lambda p_3 \beta_3 - \mu(\mu + \eta_2 + \eta_3)) + \Lambda p_2 \beta_3(\mu + \eta_2))}}{2\mu(\beta_2 - \beta_3)\beta_2} \end{aligned}$$

S_2^* is found by solving the following equation,

$$\Lambda p_2 - S_2^*\mu + \frac{(\gamma_2 - S_2^*\alpha_2)(\Lambda p_1\alpha_1 + S_2^*\mu\alpha_2 - \mu(\mu+\gamma_1+\gamma_2))}{\alpha_1(\mu+\gamma_2 - S_2^*\alpha_2)} + \frac{(\eta_2 - S_2^*\beta_2)(\Lambda p_3\beta_3 + S_2^*\mu\beta_2 - \mu(\mu+\eta_2+\eta_3))}{\beta_3(\mu+\eta_2 - S_2^*\beta_2)} = 0$$

E_0 is the party free equilibrium point, both party A and B are gone. With no parties individuals enter through constant recruitment and leave the susceptible classes through natural death, thus the form of the fixed point. This point always exists, but its stability depends on the following two equations,

$$R_A = \frac{\alpha_1\Lambda p_1 + \alpha_2\Lambda p_2}{\mu(\mu+\gamma_1+\gamma_2)} \qquad \text{and} \qquad R_B = \frac{\beta_2\Lambda p_2 + \beta_3\Lambda p_3}{\mu(\mu+\eta_1+\eta_2)}$$

These were found using the next generation matrix method [26]. When there is only party A, R_A is the basic reproduction number, defined as the number of members an individual from A will produce in a *naive* population during his lifetime. Similarly with party B. In presence of both parties, the basic reproduction number is $R_0 = \max(R_A, R_B)$. The equilibrium with only party A exists when $R_A > 1$. It is stable when the following two conditions are true,

$$R_B < 1$$

$$\frac{\frac{\mu((\bar{S_2}\mu\Lambda p_2)\alpha_1 - \Lambda p_2\alpha_2 + \mu\gamma_2)}{2\mu(\bar{S_2}\alpha_2 - \gamma_2)}}{} \pm \frac{\sqrt{(\mu((\bar{S_2}\mu\Lambda p_2)\alpha_1 - \Lambda p_2\alpha_2 + \mu\gamma_2)^2 - 4\mu^2(\bar{S_2}\mu - \Lambda p_2)(-\Lambda p_2\alpha_1\alpha_2 - \alpha_2(\gamma_2 - \bar{S_2}\alpha_2)^2 + \alpha_1(\bar{S_2}^2\alpha_2^2 - 2\bar{S_2}\alpha_2\gamma_2 + \gamma_2(\mu+\gamma_2)))}}{2\mu(\bar{S_2}\alpha_2 - \gamma_2)} < 0$$

The first condition states that the basic reproduction number of party must be less than one. The second condition is based upon how fast members leave and join party A. It is not clear when this condition will hold, Since there are members that can come from both S_1 and S_2. E_2 is the symmetric case of E_1 and has similar conditions on stability.

$$R_A < 1$$

$$\frac{\mu((S_2^{\#}\mu\Lambda p_2)\beta_3 - \Lambda p_2\beta_2 + \mu\eta_2)}{2\mu(S_2^{\#}\beta_2 - \eta_2)} \pm \frac{\sqrt{(\mu((S_2^{\#}\mu\Lambda p_2)\beta_3 - \Lambda p_2\beta_2 + \mu\eta_2)^2 - 4\mu^2(S_2^{\#}\mu - \Lambda p_2)(-\Lambda p_2\beta_3\beta_2 - \beta_2(\eta_2 - S_2^{\#}\beta_2)^2 + \beta_3((S_2^{\#})^2\beta_2^2 - 2S_2^{\#}\beta_2\eta_2 + \eta_2(\mu+\eta_2)))}}{2\mu(S_2^{\#}\beta_2 - \eta_2)} < 0$$

It also has a similar interpretation. E_3 is the co-existence of both parties. The existence and stability of this point is complicated so we not examine it directly, We note that when all three other equilibria are unstable, both $R_A, R_B > 1$.

Noticing that S_2 is found by solving a quadratic in three of the four steady states, we examine if a backward bifurcation occurs. We use the method as described by Castillo and Song [13]. First we solve for a bifurcation parameter, α_1. Using $R_0 = \max(R_A, R_B) = 1$, we assume that $R_A = 1$ and $R_B < 1$. From R_A, we solve for α_1 and find

$$\alpha_1 = \alpha_1^* = \frac{\mu(\mu+\gamma_1+\gamma_2) - \alpha_2 \Lambda p_2}{\Lambda p_1}$$

We plug this into our Jacobian matrix at the party free equilibrium,

$$\begin{pmatrix} -\mu & 0 & 0 & -\mu-\gamma_2+\frac{\Lambda p_2\alpha_2}{\mu} & 0 \\ 0 & -\mu & 0 & \gamma_2-\frac{\Lambda p_2\alpha_2}{\mu} & \eta_2-\frac{\Lambda p_2\beta_2}{\mu} \\ 0 & 0 & -\mu & 0 & \eta_3-\frac{\Lambda p_3\beta_3}{\mu} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Lambda p_2\beta_2+\Lambda p_3\beta_3-\mu(\mu+\eta_2+\eta_3)}{\mu} \end{pmatrix}$$

To use the theorem, one eigenvalue must be zero and all others must have negative real parts. Based on our assumptions this is true. The left and right eigenvectors corresponing to $\lambda = 0$ for the Jacobian matrix are as follows

$$\mathbf{v} = (0, 0, 0, v_4, 0)$$

$$\mathbf{w}^T = \left(\frac{w_4}{\mu}\left(\gamma_1 - \frac{\Lambda p_1\alpha_1}{\mu}\right), \frac{w_4}{\mu}\left(\gamma_2 - \frac{\Lambda p_2\alpha_2}{\mu}\right), 0, w_4, 0\right)$$

Both v_4 and w_4 are free. We calculate a and b to find what type of bifurcation occurs. For simplicity we let $v_4, w_4 = 1$. Since the only non-zero value of $\mathbf{v}$ is v_4 we only need to consider the fourth equation, $\frac{dA}{dt} = f_4$. We look at all combinations of partial derivatives and find that only four give non-zero amounts.

$$a = \sum_{i,j,k=1}^{5} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} = \sum_{i,j=1}^{5} v_4 w_i w_j \frac{\partial^2 f_4}{\partial x_i \partial x_j} = 2w_1\alpha_1 + 2w_2\alpha_2$$

For b we take the partial with respect to the state variables and the bifurcation parameter.

$$b = \sum_{j,k=1}^{5} v_k \frac{\partial^2 f_k}{\partial x_j \partial \alpha_1} = \sum_{j=1}^{5} v_4 \frac{\partial^2 f_4}{\partial x_j \partial \alpha_1} = v_4(A + S_1) = \frac{\Lambda p_1}{\mu}$$

Now we can classify what type of bifurcation we have at $R_0 = 1$. When $a > 0$, $b > 0$ we have a backward bifurcation. b is always greater than zero. If both

$$R_B < 1 \qquad \text{and} \qquad \gamma_1\alpha_1 + \gamma_2\alpha_2 > \frac{\Lambda p_1}{\mu}\alpha_1^2 + \frac{\Lambda p_2}{\mu}\alpha_2^2$$

we have a backward bifurcation. Since party B is losing members S_2 will have a larger proportion of individuals then normal which party A can grab. There is a balance between the rate that members join and leave that party A must satisfy.

2.2 Single Party Model With Ads

For this section we look at another simplified version of the compartmental model. The no ads model looked at how members moved to different parties. This model looks at how one party's ads can change the amount of members it can gain. The model involves two types of susceptibles, those that prefer the party and those that don't. This can be seen as a dictatorship. Individuals have one choice but are not forced to join. The television ads are acting as propaganda. An increase in advertising will provoke an increase in susceptibles moving to the preferred class. Once they move they cannot move back unless it is through becoming a party member. Individuals who cannot stand the single party system may leave the system entirely (i.e. leave the country, become politically inactive). They may only leave this way if they are in the non-preferred class. The model is defined as follows,

$$\begin{cases} \frac{dS_1}{dt} = & p_1\Lambda + \gamma_1 A + \Omega_1 T_A S_2 - \alpha_1 A S_1 - \mu S_1 \\ \frac{dS_2}{dt} = & p_2\Lambda + \gamma_2 A - \Omega_1 T_A S_2 - \alpha_2 A S_2 - \mu S_2 - \delta S_2 \\ \frac{dA}{dt} = & A(\alpha_2 S_2 + \alpha_1 S_1 - (\mu + \gamma_1 + \gamma_2)) \\ \frac{dT_A}{dt} = & r_1\left(1 - \frac{T_A}{K_1}\right) T_A \end{cases} \tag{2}$$

This model is still well-posed. Indeed the set

$$\mathcal{L} = \{(S_1, S_2, A, T_A) \in \mathbb{R}_+^4, \big| S_1 + S_2 + A \leq \frac{p_1\Lambda + p_2\Lambda}{\mu};\ T_A \leq k_1\}$$

is a compact positively invariant set of the system

All parameters have the same meaning as before but there is an extra parameter going out of S_2 to represent individuals that leave the country. We use a similar method to find the equilibrium points of the system. This system only has two susceptible classes and one infected class, but now includes an equation for the amount of ads party A is running. Both susceptible classes can be infected but those that have been influenced by ads are more susceptible. Similarly to Lemma 1, This model is well-posed.

There are two situations in which $\frac{dT_A}{dt} = 0$, when $T_A = 0$ or $T_A = K_1$. When $T_A = 0$, it is similar to the no ads model. When $T_A = K_1$, party A is running the maximum number of ads possible. The equilibrium points are the following,

$$E_0 = \left(\frac{\Lambda p_1}{\mu}, \frac{\Lambda p_2}{\delta + \mu}, 0, 0\right)$$
$$E_1 = \left(\frac{(\mu + \gamma_1 + \gamma_2) - \hat{S}_2\alpha_2}{\alpha_1}, \hat{S}_2, \frac{\Lambda p_2 - \hat{S}_2(\delta + \mu)}{\hat{S}_2\alpha_2 - \gamma_2}, 0\right)$$

where,

$$\hat{S_2} = \frac{\alpha_1(\Lambda p_1\alpha_2 + \Lambda p_2\alpha_2 + (\delta + \mu)(\gamma_2 + \mu)}{2\alpha_2(\mu\alpha_2 - \alpha_1(\delta + \mu))} \pm \frac{\sqrt{(\alpha_1(\Lambda p_1\alpha_2 + \Lambda p_2\alpha_2 + (\delta + \mu)(\gamma_2 + \mu))^2 + 4\alpha_2(\mu\alpha_2 - \alpha_1(\delta + \mu))(\Lambda p_2\alpha_1(\gamma_2 + \mu) + \gamma_2(\Lambda p_1\alpha_1 - \mu(\mu + \gamma_1 + \gamma_2)))}}{2\alpha_2(\mu\alpha_2 - \alpha_1(\delta + \mu))}$$

The party free equilibrium point with no ads, E_0, has the addition δ parameter. Note that the basic reproduction number for this system assumes party A is running maximum ads,

$$R_0 = R_A = \frac{\Lambda p_1\alpha_1(\delta + \mu + T_A\Omega_1) + \Lambda p_2(\mu\alpha_2 + T_A\alpha_1\Omega_1)}{\mu(\mu + \gamma_1 + \gamma_2)(\delta + \mu + T_A\Omega_1)}$$

We can see that if $\Omega_1 = 0$ and $\delta = 0$ then

$$R_0 = R_A = \frac{\Lambda p_1 \alpha_1 + \Lambda p_2 \alpha_2}{\mu(\mu + \gamma_1 + \gamma_2)}$$

which is the same as the previous section. Looking at E_0 we notice it always exists but its stability depends on $R_A < 1$. E_1 will exist if $R_A > 1$. This supports the first reduction of the model. Next we look at the case where $T_A = K_1$. The equilibrium points are the following,

$$E_2 = \left(\frac{\Lambda K_1 p_2 \Omega_1 + \Lambda p_1(\mu + \delta + K_1\Omega_1)}{\mu(\mu + \delta + K_1\Omega_1)}, \frac{\Lambda p_2}{\mu + \delta + K_1\Omega_1}, 0, K_1 \right)$$
$$E_3 = \left(\frac{(\mu + \gamma_1 + \gamma_2) - \tilde{S}_2\alpha_2}{\alpha_1}, \tilde{S}_2, \frac{\Lambda p_2 - \tilde{S}_2(\delta + \mu + K_1\Omega_1)}{\tilde{S}_2\alpha_2 - \gamma_2}, K_1 \right)$$

where,

$$\tilde{S_2} = \frac{\mu\alpha_2(\mu + \gamma_1 + 2\gamma_2) - \alpha_1(\Lambda p_1\alpha_2 + \Lambda p_2\alpha_2 + (\delta + \mu)(\gamma_2 + \mu) + \mu K_1\Omega_1)}{2\alpha_2(\mu\alpha_2 - \alpha_1(\delta + \mu))}$$
$$\pm \frac{\sqrt{(\mu\alpha_2(\mu + \gamma_1 + 2\gamma_2) - \alpha_1(\Lambda p_1\alpha_2 + \Lambda p_2\alpha_2 + (\delta + \mu)(\gamma_2 + \mu) + \mu K_1\Omega_1))^2 + 4\alpha_2(\mu\sigma_2 - \alpha_1(\delta + \mu))(\Lambda p_2\alpha_1(\mu + \gamma_2) + \gamma_2(\Lambda p_1\alpha_1 - \mu(\mu + \gamma_1 + \gamma_2)))}}{2\alpha_2(\mu\alpha_2 - \alpha_1(\delta + \mu))}$$

These points depend on the amount of ads the party is running. The addition of ads changes the position of the fixed points. E_2 always exists and is stable if $R_A < 1$. E_3 exists if $R_A > 1$. Like the model with no ads, stability of this point is hard to analyze. However simulations suggest it is stable if $R_0 > 1$

This model may also have a backward bifurcation. We look at a bifurcation parameter α_1 from R_A.

$$\alpha_1^* = \frac{\mu(-\Lambda p_2\alpha_2 + (\mu + \gamma_1 + \gamma_2)(\delta + \mu + K_1\Omega_1))}{\Lambda(K_1 p_2\Omega_1 + p_1(\delta + \mu + K_1\Omega_1))}$$

We assume that $T_A = K_1$ because the bifurcation may only happen if party A is running ads. We restrict the system to only the first three equations from Model 2. Looking at the Jacobian matrix at the party free equilibrium with α_1^*. This matrix has one zero eigenvalues and the other two negative. Now we can use the method for finding bifurcation again [13]. The two eigenvectors are,

$$\mathbf{v} = (0, 0, v_3)$$
$$\mathbf{w}^T = \left(\frac{w_3}{\mu}\left(\frac{\Omega_1 w_2}{w_3} + \frac{\Lambda p_2\alpha_2}{\delta + \mu + \Omega_1} - (\mu + \gamma_2) \right), \frac{w_3}{\delta + \mu + \Omega_1}\left(\gamma_2 - \frac{\Lambda p_2\alpha_2}{\delta + \mu + \Omega_1} \right), w_3 \right)$$

Using the same process as before we find,

$$a = 2(w_1\alpha_1 + w_2\alpha_2)$$
$$b = v_3\left(\frac{\Lambda K_1 p_2\Omega_1 + \Lambda p_1(\mu + \delta + K_1\Omega_1)}{\mu(\mu + \delta + K_1\Omega_1)} \right)$$

b is always positive. a is positive when the following condition is satisfied,

$$\frac{\mu\alpha_2}{\alpha_1}(\Lambda p_2\alpha_2 - \gamma_2(\delta + \mu + K_1\Omega_1)) < \Lambda p_2\alpha_2(\delta + \mu) - \left(\frac{\gamma_2(\delta + \mu)}{\delta + \mu + K_1\Omega_1} + \mu \right)$$

With this condition satisfied a backward bifurcation occurs. The most important feature of this condition it that it can be stratified by increasing the amount or effectiveness of ads. When $\delta = 0$ and $\Omega_1 = 0$, this condition is never satisfied. Ads are needed for party A to recover from poor recruitment into party A.

3 Case Study: 2008 United State Presidential Election

Following the analysis of these two reductions, we proceed to estimate our parameters. We use the 2008 presidential election as a case study. The cost of ads depends on the time of day, which channel, and amount of viewers. The most common type of commercial is thirty seconds. The cost per ad ranges from a few hundred to over thirty thousand dollars. On average one thirty second ad costs $147,000 [27]. Each candidate has a budget specifically for television ads [2].

Total spent on TV ads by Obama	$280,000,000
Total spent on TV ads by McCain	$134,000,000

Calculating K_1 and K_2 involves looking at both the budget of each candidates and the time period it was spent. The time period we examine is from January 1^{st}, 2007 to November 4^{th}, 2008, which is 673 days.

$$\begin{aligned}
K_1 \text{ (value for Obama)} \quad &= \$280,000,000/\$147,000 \\
&= 1904 \text{ commercials} \\
&= 1904/673 \text{ commercials per day} \\
K_2 \text{ (value for McCain)} \quad &= \$134,000,000/\$147,000 \\
&= 911 \text{ commercials} \\
&= 911/673 \text{ commercials per day}
\end{aligned}$$

In order to find the rate at which susceptibles switch classes, σ and Ω, first we look at the amount of each candidates budget that was devoted to television advertising and the amount of funds each candidate raise proportional to the total amount all candidates raised.

Candidate	TV Ads	Funds
Obama	.360	.670
McCain	.353	.326

Next we look at the effectiveness of each ad. Ads that influence those individuals in the middle have a higher rate then those who are in one of the extreme populations, 54.5% on those in S_2 compared to 12% on S_1 and S_3 [5]. Finally we found the total amount of ads aired during the election, 730,041 ads. We divide this by the time period and get 1085 ads per day [23]. These ads were spread across the country so we can safely assume that each person sees one ad per day. With this we can estimate the likelihood that the watched ad was from Obama or McCain was effective at changing the susceptibles preference. We get the following,

$$\begin{aligned}
\Omega_1 \text{ (Obama)} \quad &= (.360)(.670)(.545) \quad &= .1315 \\
\Omega_2 \text{ (McCain)} \quad &= (.353)(.326)(.545) \quad &= .0627 \\
\sigma_2 \text{ (Obama)} \quad &= (.360)(.670)(.120) \quad &= .0289 \\
\sigma_1 \text{ (McCain)} \quad &= (.353)(.326)(.120) \quad &= .0128
\end{aligned}$$

For births and deaths we look at the average life span able to vote. The average life span in 2008 was 78.1 years [19]. In the United States the voting age is 18. Our death rate would then be $\mu = 1/(60 * 365)$. The birth rate, Λ, is constant and estimated to be 11637 births per day [28], which agrees with $\mu N = 10297$. We assume that each new born will live the full life span, so they reach voting age at the same proportional of those just being born. The proportions, p_1, p_2, p_3, are estimated using survey data that was collected during the time of the election. Using party affiliation surveys at the beginning of the campaign trail, 2007, we find 35% identify as democrat, 28% republican, and 34% independent [20]. Based on the U.S. census data we found that there are 225.5 million voter-age individuals [12]. This amount will be split between the three susceptible classes. Political action committees, or PACs, are groups who raise money for a candidate or to spread a party's ideology. Membership of PACs range from a couple thousands to a few millions members. We use this data to estimate the amount of each parties initial population. The International Brotherhood of Electric Workers is a democratic PAC and raised the most funds during the 2008 election [24]. They had 700,000 members. The PAC that raised the most funds for McCain was The American Banker Association with 2,000,000 members including those employed by members [4].

For the rates at which individuals leave or join parties we make some assumptions. Based on the amount of party members and total number of individuals, the total rate of success of getting one individual to join one party is extremely small, $4.43 * 10^{-9}$. This value is the same for both parties but split unequally between same party parameters. This value will be split into thirds, two thirds going to α_1 and the remaining third to α_2. We repeat this for β_2 and β_3. We assume it takes one week to reach a decision to leave a party, η_2, $\gamma_2 = 1/(7 * 235500000)$. For those who leave but still have a strong preference for the party this will take twice as long, η_3, $\gamma_1 = 1/(14 * 235500000)$.

With these values we can re-examine the original model though numerical simulations. For a final table of estimates see the appendix.

4 Numerical Simulations

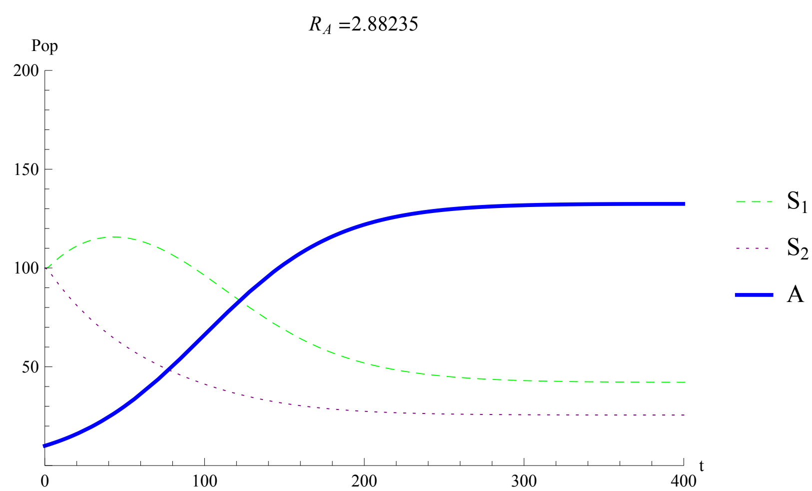


Figure 1: Single party model with ads and initial conditions $(100, 100, 10)$.

Using the original model we look at the different outcomes possible, including those with and without ads. For the stability of the single party model we look to see if there is a point at which party A exists and is stable. Figure 1 supports that in some cases this will be true. There is a large shift of members from S_2. S_1 increases, but as party A grows it slows down and eventually decreases.

In the two party system with no ads we were not able to prove existence or stability of the two party

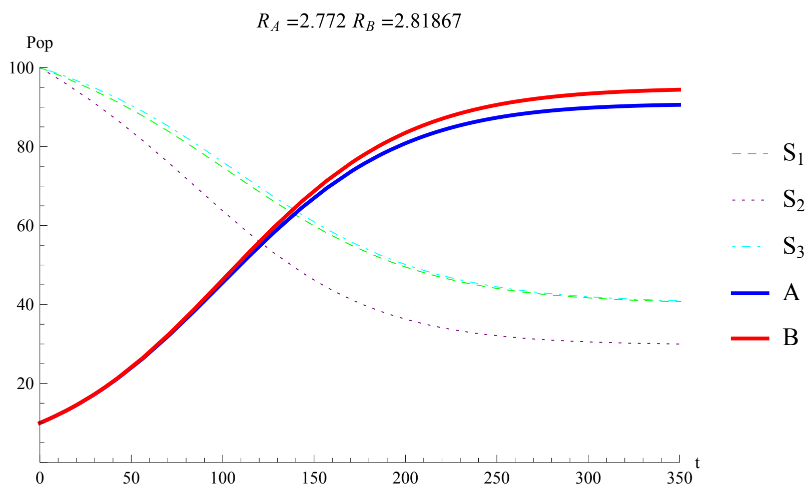


Figure 2: Model with no ads with initial conditions $(100, 100, 100, 10, 10)$. Party B and Party A stabilize

co-existence equilibrium. Through numerical simulations we were able to provide a case were both parties exist and are stable. Looking at Figure 2, this point seems to be dependent on both R_A and R_B.

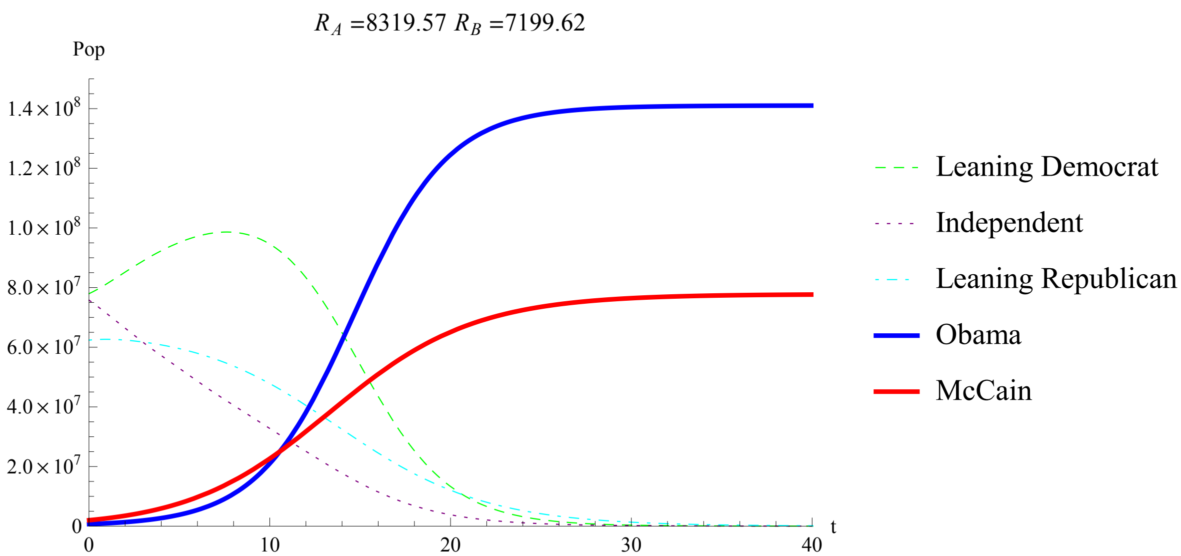


Figure 3: Model with ads using parameter estimation from 2008 presidential election, Final party size of Obama and McCain was 140 million and 70 million, respectively.

Using the parameters estimated, we can look at how the system acts under the special case of the 2008 United States presidential election. Obama started with less members but has a higher influence from ads and a larger budget. There is co-existence of both parties. The amount of members once stable is different than the amount of votes at the end of the election but Obama still won with a final member number of 140 million.

5 Conclusions

Looking at the original model and the two reductions, it is clear that advertising and budget affect the outcome of the election.

From the model with no ads we saw that in the presence of two parties there are multiple outcomes. One is a party free equilibrium where there is no party to join. As soon as someone starts a party and has a larger enough infection rate then it is sustainable. Two situations consist of only one party. In those situation if another party were to start it must have large enough recruitment rate or the first party will remain the only one. If it does overcome then the two parties will co-exist. This depends on each party's ability to infect new members. When two parties exist with no ads both parties are fighting over the middle class. From the bifurcation we can conclude that when both parties are losing members it is possible that one party begins to gain more members then they are losing and become stable. This is due to the large amount of people entering the middle class as well as the rate at which the opposing party is losing members.

The second model is more complicated. Comparing both situations in this model we can see the effect of ads on the movement of individuals. The first set of equilibrium points mirror what happened in the first reduction of the model. The involvement of ads changes the final amount of members in party A. It is interesting to note that ads do benefit the party. Even if there is no competition within the system ads increase party membership. From the bifurcation we see that if the party is at risk of dying out, losing more members than they are gaining, ads can sustain the party to some fixed point.

From Figure 1 and 2 we find that the parties approach some endemic steady state. It is not clear what the exact relationship is between stability of the points and the reproduction numbers but there is one to be found. This is true for the original and both reductions of the model.

Finally looking at the case study we find that using the estimated parameters that the number of final members is much different than reality but the results are the same and provided interesting phenomena. From the initial conditions we see that McCain had more party members but a smaller budget. Obama had less party members initially but had a much larger budget as well as a larger proportion of those that already prefer the democratic party. In the beginning both parties grow but eventually the effectiveness of the ads swayed a large proportion of those who were independent. This can been seen by the large dip in the independent class and hump in the leaning democrat class. Modeling swing voters with fixed amounts of democrats and republicans may give more accurate results.

The system can be generalized to any competing companies whose consumer base has preferences, such as Coke and Pepsi [18]. Members can be seen as those who are actively recruiting and susceptible are potential customers. This project is particular to political advertising and television.

6 Further Research

This model motivation was from interest in the different strategies used to gain political support. This model has a basis that can built off of. The addition of news media or debates would change that nature of how individuals react to single events. This model may not be able to incorporate certain types of political strategies. Alterations of this model may lead to a more general form that can accompany many types of influences.

Variation of parameters may lead to more realistic distribution of susceptibility. At the moment, there are only three classes but with this method there is an infinite amount of susceptible compartments. Ads based on this will be less effective at one end and gradually get more effective towards the other.

It would be nice to add a spatial component to the model. Members would be making physical contacts with other individuals and television ads would be able to contact large amounts of potential members while being less effective. There may be individuals that don't make many contacts and will keep their position or change their preference based on only ads or close friends.

The field of opinion dynamics has looked at voter theory. It would be interesting to see how member contact along with advertising affects the opinions of others. Also if susceptibles are able to make contact with each other then there may be another way to switch susceptible classes.

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