The impact of multiple transmission pathways and movement of vector reservoirs on the dynamics of Dengue

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Abstract

The change in geographical distribution of Dengue Fever and the introduction of new vector species in some areas are increasing the concerns of epidemiologist. It is believed that second hand tire trade and its transportation has been a factor in dengue spread as it is a mobile egg reservoir. Understanding the underlying dynamics of eggs transportation due to tire trade can help prevent or reduce dengue expansion and give alternative explanations to unexpected dengue outbreaks where vertical transmission plays an important role. In this paper, we assess the impact of transportation of tires that act as hatcheries from rural areas with dengue disease to urban areas without the disease. A mathematical model capturing dynamics of Dengue in the two distinct patches, representing rural and urban areas, is introduced and analyzed. The model takes into account the explicit movement and storage time of tires. The model was able to capture four possible situations in the field under different geographical circumstances. In the first, a new mosquito species is introduced in a naive mosquitoes species region, in the second a Dengue outbreak is generated, in the third scenario an endemic state is enhanced or induced whether urban conditions for an outbreak are met of not, in the last scenario we asses the possibility of using the second hand tire trade as a dengue control measure.

Keyword: Dengue, vertical transmission, diapause, reservoirs, tires

1 Introduction

Dengue fever is one of the most important vector-borne disease with approximately 2.5 billions people at risk of infection and 50 million dengue infections annually [18]. More than 100 tropical and subtropical countries live under endemic dengue virus conditions [11]. It is the fastest reemerging disease; which imposes an economic burden and health in affected individuals.

There are two mosquito species that can transmit the virus: *Ae. aegypti* and *Ae. albopictus*. There are four virus serotype antigenically related: DENV-1, DENV-2, DENV-3 and DENV-4 [10]. The virus is transmitted by the bite of an infected mosquito and once an individual has been infected by one serotype they are permanently immune to that serotype but only temporarily immune to the others [10, 9].

Ae. aegypti is the most common vector that transmit the disease but *Ae. albopictus*, also called Asian Tiger mosquito, is becoming an increasingly important vector because of its rapidly changing global distribution [3, 20]. The local and worldwide trade of second hand tires which often contain standing rain water and vector eggs [20] is an important dispersion cause of the mosquito species and the virus. The *Ae. albopictus* was disseminated from Asia to tropical regions worldwide and was first introduced to the Americas in the 1980s in used tires and bamboo plants shipped from Asia [3, 11]. Since then, *Ae. albopictus* has been identified in 20 countries in the Americas. Also ecologists have studied the factors that led the introduction of *Ae. albopictus* in Europe, and the conclusion was that international trade in used tires and lucky bamboo may have been main means of it introduction [16].

This conclusion is not surprising as discarded tires are considered one of the most productive containers for the mosquitoes and are frequently infected with both species [2].

Because the trade of tire abundance near human populations, the local movement of tires has become an important mobile reservoirs [22], where the spatial distribution of *Ae. aegypti* and *Ae. albopictus* in geographical zones urban-rural was studied. Moreover, larval collection from used tires may be suitable to asses rapidly the current distribution of dengue mosquitoes [14]. However, the role of movement of tires on dynamics of Dengue has been limited.

Aedes was eliminated in 1960 in almost all America. However, subsequent social and economic changes in the Americas have permitted the rapid re-infestation of the vector and the Dengue virus throughout the region. From 1960 to 1990, the annual production of tires increased from 2 to 17 million [5], suggesting that the management of their disposal and its recycling can be an important cause of dengue dispersion in the region. Another interesting case was Cuba that between 1981 and 1996 there was not dengue transmission. However, re-infestation has occurred in some areas; the municipality of Santiago de Cuba was reinfested in 1992 by *Ae. aegypti* transported in tires [15].

Diapause in the life cycle of aedes, is the phenomena that allows mosquito eggs survive during unfavorable conditions of development such as extreme winters, environmental disturbance and transportation to new geographic area could be by tire transportation. Diapause is the delay in development of eggs in response to adverse environmental conditions, i.e., when the hatcheries (tires) are left without water the eggs interrupt their development. During diapause, desiccation resistance in eggs increases due to higher concentrations of hydrocarbons at the egg surface [21].

Vertical or also called transovarial transmission is when virus is transmitted from the infected female mosquito to its eggs [8, 13, 19]. In recent studies, vertical transmission has been found in DENV-2, DENV-3 in American [13, 19] and DENV-2 from Asia [17]. This allows the mosquitoes to transmit the disease without interacting with an infected host [12, 6]. Thus vertical transmission can be responsible in some situations of the introduction of the Dengue virus to new regions due to the second hand tire commerce.

In this paper, we asses the impact of the infected tire transportation from regions with dengue virus to free vector and free dengue regions on the emergence of Dengue and possibility of an outbreak. We take into account the waiting time before processing the tires. This waiting time is important because we expect that in a well managed recycling process second hand tires immediately processed reducing the possibility of eggs eclosion. Even more, with the right regulations, this market can act as a mechanical control measure.

The mathematical model that includes a two patches (one patch representing the rural areas with presence of dengue virus and the other patch representing an urban area initially without vectors) is described in section 2. The model incorporates vertical transmission among vectors, diapause states during transportation and the waiting time before tire processing. In section 3, the details of the mathematical analysis are provided. The conditions for the following situations to occur as a consequence of tire transportation is derived: a) Introduction of a new mosquito species but no outbreak of dengue occurs, b) Introduction of a new mosquito species and possibility of a Dengue outbreak, c) An induced endemic state is generated due to the continues introduction of infected tires, and d) An enhancement of an endemicity infection due to the continuously introduction of infected tires. In section 4, a discussion of the results and explanation of the scenarios are carried out.

2 Methods

We explore the dynamics of dengue fever between humans and female mosquitoes. The modeled population is divided in rural and urban populations, where movement occurs only in the reservoirs containing eggs. We assume same the entomological parameters for both rural and urban areas. Although the similarity, the systems differ by the loss and gain of eggs from rural to urban areas. The total human population in rural area N_R and urban area N_U are constant. *Table 1* describes the meaning of the population classes.

The human susceptible class has a per-capita birth rate η, and a per-capita death rate η. Individuals in this class become infectious accordingly with the bite rate α due to infectious vectors. The vectors become infected by biting infectious hosts with a contact rate β. The rate at which humans recover from infection is γ and are permanently immune.

On the other hand, mosquitoes never recover from the disease and we only take into account the fraction κ of female mosquitoes, as they are the only ones that transmit the decease. Mosquitoes are increased due to the egg eclosion according to the development rate ω and die with a rate ϵ. Female mosquitoes oviposit eggs at a rate ϕ and the eggs die at a rate π. If the female mosquito was already infected, a fraction ν of its oviposited eggs are infected (vertical transmission). The hatcheries have a carrying capacity C_a where $a \in \{r, u\}$, and r is rural and u urban area.

The number of tires transported from rural area to urban area per unit of time is r and θ is the mean number of tires that is considered constant. Thus r/θ is the rate of eggs movement and during transportation just a fraction χ of eggs survive. $\psi(\tau_s/\tau_d)$ is

Population	Description
S_R	Number of susceptible human in rural area.
I_R	Number of infectious human in rural area.
R_R	Number of recovered human in rural area.
S_U	Number of susceptible human in urban area.
I_U	Number of infectious human in urban area.
R_U	Number of recovered human in urban area.
M_{SR}	Density of susceptible mosquitoes in rural area.
M_{IR}	Density of infectious mosquitoes in rural area.
E_{SR}	Density of susceptible eggs in rural area.
E_{IR}	Density of infectious eggs in rural area.
M_{SU}	Density of susceptible mosquitoes in urban area.
M_{IU}	Density of infectious mosquitoes in urban area.
E_{SU}	Density of susceptible eggs in urban area.
E_{IU}	Density of infectious eggs in urban area.

Table 1: State variables.

the fraction of eggs in the tires that were able to continue with their life cycle before being killed by the recycling process of the tire. Thus it is a function of the storage time τ_s before tire processing and when $\tau_s = 0$ then $\psi = 0$, while when τ_s is greater than the development time τ_s of the eggs we expect ψ be near one from below. A flow diagram of the model system is shown in *Figure 1* and also *Table 2* shows a summary of parameters.

Parameter	Description	Value	Reference
η	Per-capita birth and natural mortality rates in humans		
γ	Per-capita recovery rate	1/7	[1]
α, β	Effective biting rate, per day	$0.2 - 0.67$	[1]
C_a	Capacity carrying of hatcheries, where $a \in \{r, u\}$, and r is rural and u urban area		
ϕ	Number of eggs laid per day for every female mosquito	10	[7]
ϵ	Per-capita mortality rate of adult mosquitoes	.07	[7]
π	Per-capita mortality rate of immature stage mosquitoes	0.05	[7]
ν	Proportion of eggs that are infected by vertical transmission	$0 - 1$	
ω	Development rate of immature to mature stages	0.05	[7]
κ	Fraction of mosquitoes that are female	0.5	
$\frac{r}{\theta}$	Number of eggs per tire that are transported		
χ	Fraction of eggs that survive the transportation		
$\psi(\tau_s/\tau_d)$	Fraction of eggs in tires that were able to continue their development before tire processing.		
τ_s,τ_d	Tires storage time and egg development time, respectively.		

Table 2: Model parameters

According to the assumptions and parameters, the system differential equations that model the dynamics of dengue disease in rural area are given by:

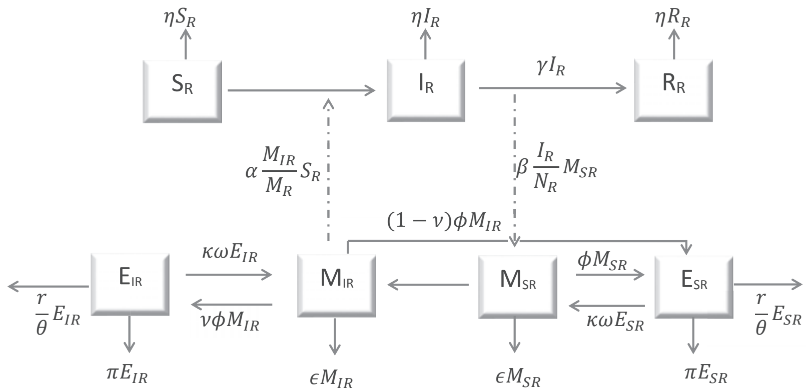

Figure 1: *Flowchart from rural dengue fever model.*

$$
\begin{aligned}
\dot{S_R} &= \eta N_R - \alpha \frac{S_R}{N_R} M_{IR} - \eta S_R, \\
\dot{I_R} &= \alpha \frac{S_R}{N_R} M_{IR} - (\eta + \gamma) I_R, \\
\dot{R_R} &= \gamma I_R - \eta R_R, \\
\dot{M_{SR}} &= \kappa\omega E_{SR} - \beta \frac{I_R}{N_R} M_{SR} - \epsilon M_{SR}, \\
\dot{M_{IR}} &= \kappa\omega E_{IR} + \beta \frac{I_R}{N_R} M_{SR} - \epsilon M_{IR}, \\
\dot{E_{SR}} &= \phi M_{SR} \left(1 - \frac{E_R}{C_r}\right) + (1-\nu)\phi M_{IR} \left(1 - \frac{E_R}{C_r}\right) - (\pi + \omega + \frac{r}{\theta}) E_{SR}, \\
\dot{E_{IR}} &= \nu\phi M_{IR} \left(1 - \frac{E_R}{C_r}\right) - (\pi + \omega + \frac{r}{\theta}) E_{IR}.
\end{aligned}
$$

The differential equations that model the dynamics of dengue disease in urban area are given by:

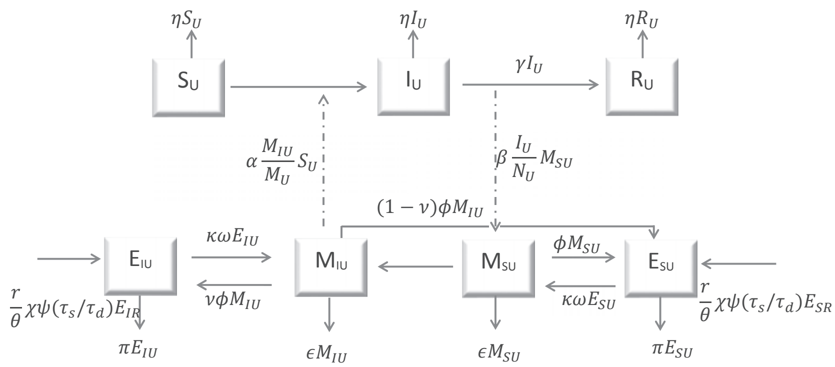

Figure 2: *Flowchart from urban dengue fever model.*

$$
\begin{aligned}
\dot{S_U} &= \eta N_U - \alpha \frac{S_U}{N_U} M_{IU} - \eta S_U, \\
\dot{I_U} &= \alpha \frac{S_U}{N_U} M_{IU} - (\eta + \gamma) I_U, \\
\dot{R_U} &= \gamma I_U - \eta R_U, \\
\dot{M_{SU}} &= \kappa\omega E_{SU} - \beta \frac{I_U}{N_U} M_{SU} - \epsilon M_{SU}, \\
\dot{M_{IU}} &= \kappa\omega E_{IU} + \beta \frac{I_U}{N_U} M_{SU} - \epsilon M_{IU}, \\
\dot{E_{SU}} &= \phi M_{SU} \left(1 - \frac{E_U}{C_u}\right) + (1-\nu)\phi M_{IU} \left(1 - \frac{E_U}{C_u}\right) - (\pi + \omega) E_{SU} + \frac{r}{\theta}\chi\psi \left(\frac{\tau_s}{\tau_d}\right) E_{SR}, \\
\dot{E_{IU}} &= \nu\phi M_{IU} \left(1 - \frac{E_U}{C_u}\right) - (\pi + \omega) E_{IU} + \frac{r}{\theta}\chi\psi \left(\frac{\tau_s}{\tau_d}\right) E_{IR}.
\end{aligned}
$$

Where $N_R = S_R + I_R + R_R$, $N_U = S_U + I_U + R_U$, $M_R = M_{SR} + M_{IR}$, $E_R = E_{SR} + E_{IR}$, $M_U = M_{SU} + M_{IU}$, $E_U = E_{SU} + E_{IU}$.

3 Analysis

The model explicitly takes into account the movement and storage time of tires. Our study focuses on four possible scenarios. The scenario I: The introduction of a mosquito species, the scenario II: A dengue outbreak emerges, the scenario III: Induced and enhance endemic states, the scenario IV: Second hand tires market as a dengue control measure and finally we present a special case: An emergence of infection in Dengue naive region by the movement of tires. In the next part we introduce each one.

3.1 The introduction of a mosquito species

The transportation of a one time batch of tires can led to the introduction of a mosquito species.

In order to obtain the conditions when this can happen, we determine the *urban offspring reproduction number (see Appendix VI)* R_M^u by means of the next generation matrix [4]. If $R_M^u > 1$, then the population of mosquitoes is able to establish itself from few eggs, while if the environment conditions make $R_M^u < 1$ the mosquito population will eventually become extinct. On the other hand, in a batch of N_T tires, the number of eggs that arrive alive to the urban area is given by

$$\frac{\omega}{\pi+\omega}\frac{N_T}{\theta}\chi\psi\left(\frac{\tau_s}{\tau_d}\right)E_R^* = \frac{N_T}{\theta}\chi\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{R_M^r - 1}{R_M^r}C_r$$

where $\frac{\omega}{\pi+\omega}$ is the probability of an egg hatching into a mosquito before dying of natural causes and E_R^* is the stationary number of eggs in the rural area, R_M^r is the *rural offspring reproduction number (see Appendix V)* and $\chi\psi(\frac{\tau_s}{\tau_d})$ is the fraction of eggs that survive before the tire processing cycle is made. Thus the introduction of a mosquito species will occur when the following conditions are meet,

$$\begin{aligned} R_M^u = \frac{\kappa\omega\phi}{\epsilon(\pi+\omega)} &> 1 \\ \textit{and } \frac{\omega}{\pi+\omega}\frac{N_T}{\theta}\chi\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{R_M^r - 1}{R_M^r}C_r &> 1 \end{aligned} \tag{1}$$

If the tire recycling becomes an established market with a constant flux of tires from a rural to an urban area, then the expected waiting time T_M before the introduction of a mosquito species from the rural to the urban population is given by the inverse of the rate of eggs introduction

$$T_M = \left[\frac{r}{\theta}\chi\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{R_M^r - 1}{R_M^r}C_r\right]^{-1} \qquad \text{for} \qquad R_M^r > 1 \tag{2}$$

3.2 A dengue outbreak emerges

If there is continuous introduction of tires to the urban area from a rural area where Dengue is endemic, then there might be possibility of a dengue outbreak via transported infected eggs. In order for this to happen the conditions (2) must be met, but also the *basic reproductive number without vertical transmission* in the urban area $R_0^u > 1$. Its value is given by $R_0^u = \sqrt{\frac{\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}}$ *(see Appendix III)*, where we have neglected vertical transmission as its effect very small at the beginning of an outbreak. Thus

$$\sqrt{\frac{\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}} > 1 \tag{3}$$

in addition to (2) should be meet.

In a continuous importation practices from an endemic region, the waiting time before an outbreak T_o is given by the inverse of the effective infected female mosquito introduction and is given by

$$T_o = \left[\frac{\kappa\omega}{\pi + \omega} \frac{r}{\theta} \chi\psi \left(\frac{\tau_s}{\tau_d} \right) E_{IR}^* \right]^{-1} \tag{4}$$

where $\frac{r}{\theta}\chi\psi\left(\frac{\tau_s}{\tau_d}\right)$ is the fraction of successful imported eggs, $\frac{\kappa\omega}{\pi+\omega}$ is the probability of an egg hatching into a female mosquito before dying of natural causes, and E_{IR}^* is the number of infected eggs in the rural area where dengue is endemic.

In case of the introduction of just one batch of N_T tires, in addition to (2) and (3) the following condition

$$\frac{\kappa\omega}{\pi + \omega} \frac{N_T}{\theta} \chi\psi \left(\frac{\tau_s}{\tau_d} \right) E_{IR}^* > 1 \tag{5}$$

should meet, i.e. effectively more than 1 infected female mosquito should be introduced.

3.3 Induced and enhance endemic states

There could be a situation when $R_0^u < 1$ but the continuous introduction of infected eggs in tires coming from a dengue endemic rural area can induce a dengue endemic state in the urban area. This endemic state is not maintained by the intrinsic dynamics of the disease in the urban area and should cease if the introduction of infected eggs is interrupted *see Figures 3 and 4*. In this situation, the expected number of dengue cases at any given time is

$$I_U^* = \frac{M_{IU}^*}{M_{IU}^* - N_U \frac{\eta}{\alpha}} \tag{6}$$

where M_{IU}^* is the stationary population of infected mosquitoes and its value can be found as the solution of the following quadratic equation for it,

$$\begin{aligned}
&(R_o^u)^2(\eta+\gamma)(\frac{\eta+\gamma}{\beta} + N_u)(M_{IU}^*)^2 \\
&-(\eta+\gamma)\left((R_o^u)^2(\frac{\kappa\omega}{\beta} E_{IU}^* + N_U M_U) - \eta N_U \right) M_{IU}^* \\
&\qquad\qquad - \frac{\kappa\omega}{\epsilon} \eta N_U E_{IU}^* = 0
\end{aligned} \tag{7}$$

where E_{IU}^* is the stationary number of infected eggs in the urban area and is given by

$$E_{IU}^* = \frac{r\chi\psi(\frac{\tau_s}{\tau_d})}{\theta(\omega+\pi)} E_{IR}^* \tag{8}$$

In cases where dengue is already endemic in the urban area, the continuous importation of tires can enhance the number of infected people *see Figures 3 and 4*. The number of infections at any given time is given by (6) when $R_0^u > 1$ in (7).

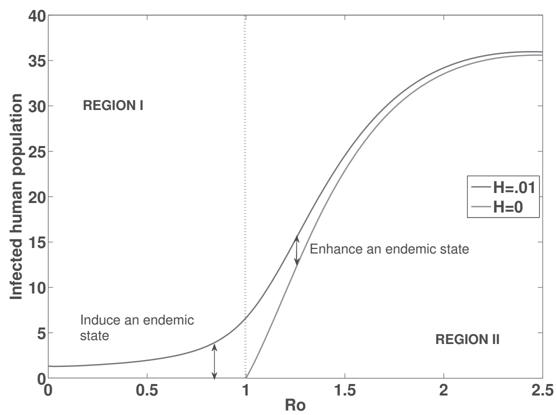

Figure 3: *It is possible induce an endemic state even though* $R_0^u < 1$ *(Region I) if there is a continuous flow of tires from endemic rural area. On the other hand, if the disease already exists, tires transport with eggs only enhance the endemic state (Region II).*

3.4 Second hand tire market as a dengue control measure

If tires are processed immediately or at least soon after arrival to urban area, i.e. $\tau_s << \tau_d$ the introduction of dengue fever it is not possible. This is because the eggs die during tire processing before hatching. On the other hand in the rural area the diminishing of eggs reduce *rural reproduction number (see Appendix I)* R_0^r and thus the number of dengue cases get reduced. In order for this happen, the withdrawal of eggs in the rural area should change offspring number from $R_M^r > 1$ to $R_M^r < 1$. If this does not happen there will be always the risk of introducing infected eggs into the urban area unless the tires were processed immediately $\tau_s/\tau_d << 1$. We can estimate the maximum storage time τ_s to minimize the risk of dengue dispersion with the following condition. The waiting time before an effective introduction of an infected female mosquito (4) should be larger than the extinction time of the vectors in the rural area. We can use the Jacobian matrix when $R_M^r < 1$ of the vector demography (10), to obtain an estimation of its extinction time. The inverse of the smallest absolute value from its eigenvalues is an estimator of its extinction time. Thus, the following condition should meet to reduce the risk of dengue dispersion in an established marked where tires are continuously imported

$$T_o > \left|\frac{1}{2}(\gamma + \sqrt{\xi})\right|^{-1} \quad \text{and} \quad R_M^r = \frac{\kappa\omega\phi}{\epsilon(\pi + \omega + r/\theta)} < 1 \tag{9}$$

where $\gamma = -(\epsilon+\pi+\omega+r/\theta)$, $\xi = \gamma^2 - 4\Xi$ and $\Xi = \epsilon(\pi+\omega+r/\theta)(1-R_M^r)$ (see appendix V), and at the same time this works as a control measure in the rural area.

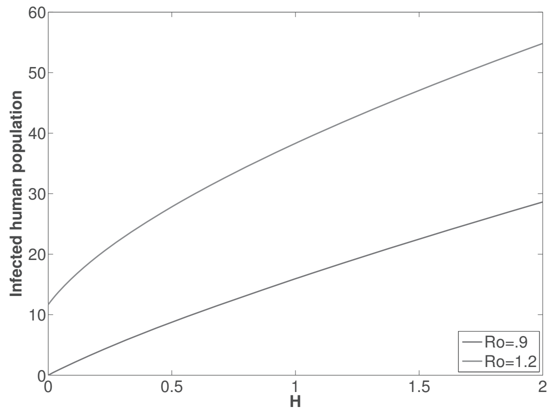

Figure 4: *Dengue cases as the number of introduced infected eggs per unit time H are increased. Red line $R_0^u > 1$ and blue line $R_0^u < 1$.*

3.5 An emergence of infection in Dengue naive region by the movement of tires.

Finally we can calculate the secondary human infections in the urban disease free area caused by the human infections in the rural area at the beginning of an outbreak $R_{r\to u}$ *(see appendix IV)* to asses the impact of second hand tire market in an outbreak. There will be one initial case of dengue virus in the urban area related with tire transportation for each $1/R_{r\to u}$ cases in the rural area, where

$$R_{r\to u} = \frac{\alpha\kappa\omega r\chi}{\epsilon(\omega+\pi)(\theta(\omega+\pi)+r)}\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{\nu\phi}{\epsilon}\frac{\beta}{(\eta+\gamma)}.$$

4 Discussion

The trade of second-hand tires in tropical and subtropical areas, influence the spread of dengue disease because the discarded tires are considered one of the most productive Aedes hatcheries [2]. For this reason, tires containing infected eggs that may enter a diapause state are a possible mechanism in the spread of dengue. The introduction of Aedes to a naive region is even more probable because it adapts well to a wide range of environmental conditions.

As a result of vertical transmission, the Aedes mosquito can transmit the disease to their offspring and therefore a new generation of infected vectors will transmit the disease without the need to interact with infected individuals.

The analysis of the presented model explicitly takes into account the movement and storage time of tires. Our study focuses on four possible scenarios. In the first one, the transport of a batch of tires to a zone free of the disease led to the introduction of the mosquito species. This happens when the environmental conditions are appropriate, thus $R_M^u > 1$. The expected waiting time before the successful introduction of the species is inversely proportional to the rate of tire transportation. Further more, vertical transmission led to infected eggs and thus the dengue virus can be introduced to regions free of the disease. If the geographical conditions of an urban area where tires are being imported are so that $R_0^u > 1$, an outbreak can be generated in the urban area or an already endemic state enhance. In the case of a naive dengue region, the storage time before the processing of tires plays an important role in the waiting time before an outbreak

If it happens that $R_0^u < 1$ then there can not be an outbreak but some infections are constantly expected while the introduction of infected eggs continues. This is what we called an induced endemic state and it should cease if the importation of tires is stopped or better regulated.

It is possible to reduce the likelihood of a dengue outbreak if the appropriate authorities regulate the processing time of the tires. This is because if the tires are processed immediately upon reaching the urban area $\tau_s << \tau_d$ the introduction of dengue fever it is not possible. On the other hand, in the rural area the movement of tires works like a measure of control. The movement of tires reduces the density of eggs in the rural area R_M^r and then the number of dengue cases get reduced R_0^r. Finally, our model allows the estimation of the number of secondary human infections in the urban area caused by the human infections in the rural area, allowing the quantification of the impact of tire market in dengue dispersal.

The introduction of a new species is independent of vertical transmission, on the other hand the spread of the disease is closely related to vertical transmission, diapause and hatcheries movement (tires).

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Appendix I: Basic reproduction number in rural area

We calculated the basic reproductive number using the next generation matrix method $\dot{\mathfrak{X}} = \mathfrak{F} - \mathfrak{V}$ [4]. The infected classes on rural model are: I_R, E_{IR} and M_{IR}, then, the information is separated into two matrices, the first one corresponds to new infection and the second one corresponds to disease progression, that is

$$\dot{\mathfrak{X}} = \begin{pmatrix} \dot{I} \\ \dot{E_{IR}} \\ \dot{M_{IR}} \end{pmatrix}$$

$$\mathfrak{F} = \begin{pmatrix} \alpha \frac{S_R}{N_R} M_{IR} \\ \nu\phi M_{IR}\left(1 - \frac{E_R}{C_r}\right) \\ \beta \frac{I_R}{N_R} M_{SR} \end{pmatrix} \qquad \mathfrak{V} = \begin{pmatrix} (\eta+\gamma) I_{IR} \\ (\pi + \omega + \frac{r}{\theta}) E_{IR} \\ \epsilon M_{IR} - \kappa\omega E_{IR} \end{pmatrix}$$

The Jacobian matrices are:

$$F = \begin{pmatrix} 0 & 0 & \alpha \frac{S_R}{N_R} \\ 0 & -\frac{\nu\phi M_{IR}}{C_r} & \nu\phi\left(1 - \frac{E_R}{C_r}\right) \\ \beta \frac{M_{SR}}{N_R} & 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} (\eta+\gamma) & 0 & 0 \\ 0 & (\pi+\omega) + \frac{r}{\theta} & 0 \\ 0 & -\kappa\omega & \epsilon \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\eta+\gamma)} & 0 & 0 \\ 0 & \frac{\theta}{\theta(\pi+\omega)+r} & 0 \\ 0 & \frac{\theta\kappa\omega}{\epsilon(\theta(\pi+\omega)+r)} & \frac{1}{\epsilon} \end{pmatrix}$$

Then we evaluated the Jacobian matrices at the disease free equilibrium $S_R = N_R, M_{SR} = M_R,\ E_{SR} = E_R,\ I_R = M_{IR} = E_{IR} = 0$. Then we found the eigenvalues of $K = FV^{-1}$ since the basic reproductive number is the spectral radius, then we need the maximum of the eigenvalues of K, this will be the basic reproductive number.

$$K = \begin{pmatrix} 0 & \frac{\alpha\theta\kappa\omega}{\epsilon(\theta(\pi+\omega)+r)} & \frac{\alpha}{\epsilon} \\ 0 & \frac{\nu\phi\theta\kappa\omega}{\epsilon(\theta(\pi+\omega)+r)}\left(1-\frac{E}{C_r}\right) & \frac{\nu\phi}{\epsilon}\left(1-\frac{E}{C_r}\right) \\ \frac{\beta N}{M^*\eta+\gamma} & 0 & 0 \end{pmatrix}$$

There are three eigenvalues, one of them is zero, the other is smaller, so the maximum is

$$R_0^r = \frac{1}{2}\frac{\nu\phi\kappa\omega\theta}{\epsilon(\theta(\pi+\omega)+r)}\left(1-\frac{E_R^*}{C_r}\right) + \frac{1}{2}\sqrt{\left(\frac{\nu\phi\kappa\omega\theta}{\epsilon(\theta(\pi+\omega)+r)}\left(1-\frac{E_R^*}{C_r}\right)\right)^2 + \frac{4\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}}$$

Appendix II: Basic reproduction number in urban area

Following the above idea, we calculate the urban reproductive number. The infected classes on urban model are: I_U, E_{IU} and M_{IU}, so the matrices $\mathfrak{F}$ and $\mathfrak{V}$ take the following shape.

$$\mathfrak{F} = \begin{pmatrix} \alpha\frac{S_U}{N_U}M_{IU} \\ \nu\phi M_{IU}\left(1-\frac{E_U}{C_u}\right) \\ \beta\frac{I_U}{N_U}M_{SU} \end{pmatrix} \qquad \mathfrak{V} = \begin{pmatrix} (\eta+\gamma)I_{IU} \\ (\pi+\omega)E_{IU} - \frac{r}{\theta}\chi\psi(\tau_s/\tau_d)E_{IR} \\ \epsilon M_{IU} - \kappa\omega E_{IU} \end{pmatrix}$$

The Jacobian matrices are:

$$F = \begin{pmatrix} 0 & 0 & \alpha\frac{S_U}{N_U} \\ 0 & -\frac{\nu\phi M_{IU}}{C_u} & \nu\phi\left(1-\frac{E_U}{C_u}\right) \\ \beta\frac{M_{SU}}{N_U} & 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} (\eta+\gamma) & 0 & 0 \\ 0 & (\pi+\omega) & 0 \\ 0 & -\kappa\omega & \epsilon \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\eta+\gamma)} & 0 & 0 \\ 0 & \frac{1}{\pi+\omega} & 0 \\ 0 & \frac{\kappa\omega}{\epsilon(\pi+\omega)} & \frac{1}{\epsilon} \end{pmatrix}$$

Then we evaluated the Jacobian matrices at the disease free equilibrium $S_U = N_U, M_{SU} = M_U,\ E_{SU} = E_U,\ I_U = M_{IU} = E_{IU} = 0$. Then we found the eigenvalues of $K = FV^{-1}$ since the basic reproductive number is the spectral radius, then we need the maximum of the eigenvalues of K, this will be the basic reproductive number.

$$K = \begin{pmatrix} 0 & \frac{\alpha\kappa\omega}{\epsilon(\pi+\omega)} & \frac{\alpha}{\epsilon} \\ 0 & \frac{\nu\phi\kappa\omega}{\epsilon(\pi+\omega)}\left(1-\frac{E_U}{C_u}\right) & \frac{\nu\phi}{\epsilon}\left(1-\frac{E_U}{C_u}\right) \\ \frac{\beta N}{(\eta+\gamma)M^*} & 0 & 0 \end{pmatrix}$$

There are three eigenvalues, one of them is zero, the other is smaller, so the maximum is

$$R_0^u = \frac{1}{2}\frac{\nu\phi\kappa\omega}{\epsilon(\pi+\omega)}\left(1-\frac{E_U^*}{C_u}\right) + \frac{1}{2}\sqrt{\left(\frac{\nu\phi\kappa\omega}{\epsilon(\pi+\omega)}\left(1-\frac{E_U^*}{C_u}\right)\right)^2 + \frac{4\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}}$$

The basic reproductive number of the complete model is the maximum of the two reproductive numbers, *rural reproduction number* and *urban reproduction number*.

$$R_0 = \max\{R_0^r,\ R_0^u\}$$

Appendix III: Basic reproduction number without vertical transmission in urban area

Following the same idea, we calculate the urban reproductive number. The infected classes on urban model are: I_U, E_{IU} and M_{IU}, so the matrices $\mathfrak{F}$ and $\mathfrak{V}$ take the following shape.

$$\mathfrak{F} = \begin{pmatrix} \alpha\frac{S_U}{N_U}M_{IU} \\ 0 \\ \beta\frac{I_U}{N_U}M_{SU} \end{pmatrix} \qquad \mathfrak{V} = \begin{pmatrix} (\eta+\gamma)I_{IU} \\ (\pi+\omega)E_{IU} - \frac{r}{\theta}\chi\psi(\tau_s/\tau_d)E_{IR} \\ \epsilon M_{IU} - \kappa\omega E_{IU} \end{pmatrix}$$

The Jacobian matrices are:

$$F = \begin{pmatrix} 0 & 0 & \alpha\frac{S_U}{N_U} \\ 0 & 0 & 0 \\ \beta\frac{M_{SU}}{N_U} & 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} (\eta+\gamma) & 0 & 0 \\ 0 & (\pi+\omega) & 0 \\ 0 & -\kappa\omega & \epsilon \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\eta+\gamma)} & 0 & 0 \\ 0 & \frac{1}{\pi+\omega} & 0 \\ 0 & \frac{\kappa\omega}{\epsilon(\pi+\omega)} & \frac{1}{\epsilon} \end{pmatrix}$$

Then we evaluated the Jacobian matrices at the disease free equilibrium $S_U = N_U, M_{SU} = M_U,\ E_{SU} = E_U,\ I_U = M_{IU} = E_{IU} = 0$. Then we found the eigenvalues of $K = FV^{-1}$ since the basic reproductive number is the spectral radius, then we need the maximum of the eigenvalues of K, this will be the basic reproductive number.

$$K = \begin{pmatrix} 0 & \frac{\alpha\kappa\omega}{\epsilon(\pi+\omega)} & \frac{\alpha}{\epsilon} \\ 0 & 0 & 0 \\ \frac{\beta N}{(\eta+\gamma)M^*} & 0 & 0 \end{pmatrix}$$

There are three eigenvalues, one of them is zero, the other is smaller, so the maximum is

$$R_0^u = \sqrt{\frac{\beta N}{(\mu+\gamma)M^*}\frac{\alpha}{\epsilon}}$$

Appendix IV: Number of transmissions from rural to urban area

To find transmission from rural to urban reproduction number, we following the same idea to calculate the above basic reproduction numbers, so we want to know how many infections could cause an individual of the rural population in the urban population by the movement of infected tires, for that we are consider that in rural population the disease is endemic. The infected classes on full model are: $I_R,\ E_{IR},\ M_{IR},\ I_U,\ E_{IU},\ M_{IU}$. The information is separated into two matrices, the first one corresponds to new infection $\mathfrak{F}$ and the second one corresponds to disease progression $\mathfrak{V}$, that is

$$\mathfrak{F} = \begin{pmatrix} \alpha\frac{S_U}{N_U}M_{IU} \\ \nu\phi M_{IU}\left(1-\frac{E_U}{C_u}\right) \\ \beta\frac{I_U}{N_U}M_{SU} \\ \alpha\frac{S_R}{M_R}M_{IR} \\ \nu\phi M_{IR}\left(1-\frac{E_R}{C_r}\right) \\ \beta\frac{I_R}{N_R}M_{SR} \end{pmatrix} \qquad \mathfrak{V} = \begin{pmatrix} (\eta+\gamma)I_{IU} \\ (\pi+\omega)E_{IU} - \frac{r}{\theta}\chi\psi(\tau_s/\tau_d)E_{IR} \\ \epsilon M_{IU} - \kappa\omega E_{IU} \\ (\eta+\gamma)I_{IR} \\ (\pi+\omega)E_{IR} + \frac{r}{\theta}E_{IR} \\ \epsilon M_{IR} - \kappa\omega E_{IR} \end{pmatrix}$$

The Jacobian matrices are:

$$F = \begin{pmatrix} 0 & \alpha\frac{S_U}{N_U} & 0 & 0 & 0 & 0 \\ \beta\frac{M_{SU}}{N_U} & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu\phi\left(1-\frac{E_U}{C_u}\right) & -\frac{\nu\phi M_{IU}}{C_u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha\frac{S_R}{N_R} & 0 \\ 0 & 0 & 0 & \beta\frac{M_{SR}}{N_R} & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu\phi\left(1-\frac{E_R}{C_r}\right) & -\frac{\nu\phi M_{IR}}{C_r} \end{pmatrix}$$

$$V = \begin{pmatrix} (\eta+\gamma) & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\kappa\omega & 0 & 0 & 0 \\ 0 & 0 & (\pi+\omega) & 0 & 0 & -\frac{r}{\theta}\chi\psi(\tau_s/\tau_d) \\ 0 & 0 & 0 & (\eta+\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon & -\kappa\omega \\ 0 & 0 & 0 & 0 & 0 & (\pi+\omega+\frac{r}{\theta}) \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\eta+\gamma)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & \frac{\kappa\omega}{\epsilon(\pi+\omega)} & 0 & 0 & \frac{\kappa\omega r\chi}{\epsilon(\pi+\omega)(\theta(\pi+\omega)+r)}\psi\left(\frac{\tau_s}{\tau_d}\right) \\ 0 & 0 & \frac{1}{(\pi+\omega)} & 0 & 0 & \frac{r\chi}{(\pi+\omega)(\theta(\pi+\omega)+r)}\psi\left(\frac{\tau_s}{\tau_d}\right) \\ 0 & 0 & 0 & \frac{1}{(\eta+\gamma)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\epsilon} & \frac{\kappa\omega\theta}{\epsilon(\theta(\pi+\omega)+r)} \\ 0 & 0 & 0 & 0 & 0 & \frac{\theta}{\theta(\pi+\omega+r)} \end{pmatrix}$$

Then we evaluated the Jacobian matrices at the disease free equilibrium. Then we found $K = FV^{-1}$. To get the *number of transmissions from rural to urban area* we obtain K^3 with this matrix in the column if infectious rural population we get the following.

$$R_{r\to u} = \frac{\alpha\kappa\omega r\chi}{\epsilon(\omega+\pi)(\theta(\omega+\pi)+r)}\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{\nu\phi}{\epsilon}\frac{\beta}{(\eta+\gamma)}$$

Appendix V: Vector demography

Considering the last one four equation of rural model and doing $\dot{M_{SR}}+\dot{M_{IR}}$ and $\dot{E_{SR}}+\dot{E_{IR}}$ it is obtained.

$$
\begin{aligned}
\dot{M} &= \kappa\omega E - \epsilon M, \\
\dot{E} &= \phi M - (\pi + \omega + \frac{r}{\theta})E.
\end{aligned} \tag{10}
$$

We calculate the basic offspring number of rural mosquitoes, using the method [4] we write the system(10) as $\dot{\mathfrak{X}} = \mathfrak{F} - \mathfrak{V}$.

$$
\dot{\mathfrak{x}} = \begin{pmatrix} \dot{M} \\ \\ \dot{E} \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} \kappa\omega E \\ \\ 0 \end{pmatrix}, \quad \mathfrak{V} = \begin{pmatrix} \epsilon M \\ \\ (\pi + \omega + \frac{r}{\theta})E - \phi M \left(1 - \frac{E}{C}\right) \end{pmatrix}.
$$

The Jacobian matrices F and V, associated with $\mathfrak{F}$ and $\mathfrak{V}$ respectively, at the vector free equilibrium $M^* = 0$, $E^* = 0$ are.

$$
F = \begin{pmatrix} 0 & \kappa\omega \\ \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \epsilon & 0 \\ \\ -\phi & (\pi + \omega + \frac{r}{\theta}) \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ \\ \frac{\phi}{\epsilon(\pi+\omega+\frac{r}{\theta})} & \frac{1}{\pi+\omega+\frac{r}{\theta}} \end{pmatrix},
$$

$$
K = FV^{-1} = \begin{pmatrix} \frac{\kappa\omega\phi}{\epsilon(\pi+\omega+\frac{r}{\theta})} & \frac{\kappa\omega}{\pi+\omega+\frac{r}{\theta}} \\ \\ 0 & 0 \end{pmatrix}.
$$

The eigenvalues of K are 0 and $\frac{\kappa\omega\phi}{\epsilon(\pi+\omega+\frac{r}{\theta})}$, so the rural offspring reproduction number is given by:

$$
R_M^r = \frac{\kappa\omega\phi}{\epsilon(\pi + \omega + \frac{r}{\theta})}
$$

The system has two stationary states $E^* = M^* = 0$ and $M^* = \frac{C\kappa\omega}{\epsilon}\left(\frac{R_M^r - 1}{R_M^r}\right)$, $E^* = C\left(\frac{R_M^r - 1}{R_M^r}\right)$.

The linearizing around the trivial stationary solutions is done. For this, we will calculate the Jacobian matrix around the equilibrium point $(0, 0)$.

$$
\mathbf{DF(0,0)} = \begin{pmatrix} -\epsilon & \kappa\omega \\ \\ \phi & -(\pi + \omega + \frac{r}{\theta}) \end{pmatrix}.
$$

We get the following characteristic polynomial,

$$\lambda^2 + (\epsilon + \pi + \omega + \frac{r}{\theta})\lambda + \epsilon(\pi + \omega + \frac{r}{\theta})(1 - R_M) = 0 \tag{11}$$

whose roots are of the shape

$$\lambda_{\pm} = \frac{1}{2}(\gamma \pm \sqrt{\xi})$$

where $\gamma = -(\epsilon + \pi + \omega + \frac{r}{\theta})$, $\xi = \gamma^2 - 4\Xi$ and $\Xi = \epsilon(\pi + \omega + \frac{r}{\theta})(1 - R_M)$.

Appendix VI: Urban offspring reproduction number

Considering the last one four equation of urban model and doing $\dot{M_{SU}} + \dot{M_{IU}}$ and $\dot{E_{SU}} + \dot{E_{IU}}$ it is obtained.

$$\begin{aligned} \dot{M} &= \kappa\omega E - \epsilon M, \\ \dot{E} &= \phi M - (\pi + \omega)E + \frac{r\chi}{\theta}\psi\left(\frac{\tau_s}{\tau_d}\right)E_R. \end{aligned} \tag{12}$$

We calculate the urban offspring number, using the method [4] we write the system(12) as $\dot{\mathfrak{X}} = \mathfrak{F} - \mathfrak{V}$.

$$\dot{\mathfrak{x}} = \begin{pmatrix} \dot{M} \\ \\ \dot{E} \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} \kappa\omega E \\ \\ 0 \end{pmatrix}, \quad \mathfrak{V} = \begin{pmatrix} \epsilon M \\ \\ (\pi + \omega)E - \phi M\left(1 - \frac{E}{C}\right) - \frac{r\chi}{\theta}\psi\left(\frac{\tau_s}{\tau_d}\right)E_R \end{pmatrix}.$$

The Jacobian matrices F and V, associated with $\mathfrak{F}$ and $\mathfrak{V}$ respectively, at the vector free equilibrium $M^* = 0$, $E^* = 0$ are.

$$F = \begin{pmatrix} 0 & \kappa\omega \\ \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \epsilon & 0 \\ \\ -\phi & (\pi + \omega) \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ \\ \frac{\phi}{\epsilon(\pi+\omega)} & \frac{1}{\pi+\omega} \end{pmatrix},$$

$$K = FV^{-1} = \begin{pmatrix} \frac{\kappa\omega\phi}{\epsilon(\pi+\omega)} & \frac{\kappa\omega}{\pi+\omega} \\ \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of K are 0 and $\frac{\kappa\omega\phi}{\epsilon(\pi+\omega)}$, so the urban offspring reproduction number is given by:

$$R_M^u = \frac{\kappa\omega\phi}{\epsilon(\pi + \omega)}$$

Appendix A: Summary of reproduction number

- Urban offspring reproduction number.

$$R_M^u = \frac{\kappa\omega\phi}{\epsilon(\pi+\omega)}$$

- Rural offspring reproduction number.

$$R_M^r = \frac{\kappa\omega\phi}{\epsilon(\pi+\omega+\frac{r}{\theta})}$$

- Basic reproduction number without vertical transmission in urban area.

$$R_0^u = \sqrt{\frac{\beta N}{(\mu+\gamma)M^*}\frac{\alpha}{\epsilon}}$$

- Basic reproduction number in urban area.

$$R_0^u = \frac{1}{2}\frac{\nu\phi\kappa\omega}{\epsilon(\pi+\omega)}\left(1-\frac{E_U^*}{C_u}\right) + \frac{1}{2}\sqrt{\left(\frac{\nu\phi\kappa\omega}{\epsilon(\pi+\omega)}\left(1-\frac{E_U^*}{C_u}\right)\right)^2 + \frac{4\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}}$$

- Basic reproduction number in rural area.

$$R_0^r = \frac{1}{2}\frac{\nu\phi\kappa\omega\theta}{\epsilon(\theta(\pi+\omega)+r)}\left(1-\frac{E_R^*}{C_r}\right) + \frac{1}{2}\sqrt{\left(\frac{\nu\phi\kappa\omega\theta}{\epsilon(\theta(\pi+\omega)+r)}\left(1-\frac{E_R^*}{C_r}\right)\right)^2 + \frac{4\alpha}{\epsilon}\frac{\beta N}{(\eta+\gamma)M^*}}$$

- Number of transmissions from rural to urban area.

$$R_{r\to u} = \frac{\alpha\kappa\omega r\chi}{\epsilon(\omega+\pi)(\theta(\omega+\pi)+r)}\psi\left(\frac{\tau_s}{\tau_d}\right)\frac{\nu\phi}{\epsilon}\frac{\beta}{(\eta+\gamma)}$$