# Minimizing recidivism by optimizing profit: a theoretical case study of incentivized reform in a Louisiana prison

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## Abstract

Recidivism is the phenomenon where an individual returns to criminal activity after being released from prison. Many prisoners in the U.S. end up back in jail within 5 years. Using Louisiana as a case study, we show that prison management can minimize recidivism by subsidizing reform programs in for-profit prisons. Accounting for such an incentive program allows us to observe alterations in prison profit optimization. Within the model, the prison alters the proportion of time that each inmate spends in the reform program. The incarceration dynamics respond to the average proportion of time that prisoners spend in reform. We determine that the prison's profit is the most sensitive to the value of the incentive, the fixed cost per prisoner, the effectiveness of the instated reform program, the number of first time offenders currently in the prison, and the per diem rate per prisoner the prison receives from the state. Prisons with higher initial incomes require a larger incentive to obtain the same results as their less profitable neighbors. The reduction in recidivism has diminishing returns as the incentive is increased.

**Keywords**: recidivism, incarceration, Louisiana, adaptive system, cost-benefit analysis, discrete time model

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# 1   Introduction

The United States has the second highest incarceration rate in the world at 698 inmates per 100,000 individuals, behind the small island nation of Seychelles.[1] Within the last 35 years, the prison population has accumulated more than a million incarcerated individuals.[2,3] The trend of increasing incarceration rates goes back to June of 1971 when President Richard Nixon identified drug abuse as a national threat and officially declared a "war on drugs".[4] Prison populations rose as sentence lengths increased with new mandated minimum sentences for controlled substances.[5,6] According to a study conducted in 2011 by the Bureau of Prisons, 39% of prisons within the United States were overcrowded as a result of this population expansion.[7]

The state of Louisiana has an extremely high rate of crime (4,101 crimes per 100,000 individuals) and the highest rate of incarceration in the nation (847 prisoners per 100,000 residents), both statistics well above the national average.[2,8] In the 1990's, the issue surrounding overcrowded prisons became increasingly problematic for the 12 state prisons to function efficiently. As a result of budget constraints, the government encouraged individual parishes to develop localized prisons that the local sheriff would oversee. This led to the development of around 160 local jails throughout the state.[9,10] The sheer volume of crime and incarceration has led to the creation of a plethora of job opportunities in correctional facilities. The Louisiana Budget Project reported that "Total government [Full Time Employees employed in prisons] per capita data suggests that Louisiana ranks relatively high – 7th among the states" in the per-capita rate of full-time prison employees.[11]

Local and state prisons have different demographics in regards to the type of crime committed by race, sentence length, etc. About 25% of state inmates are serving fixed terms or life sentences compared to only 3.5% of inmates in local prisons. A similar disparity can be observed amongst non-violent criminals as they comprise 81.7% of all inmates in local prisons compared to 36.4% of those in state prisons. In addition, the average age of local prisoners is 5 years below that of those in state prisons.[12] A recent Bureau of Justice Statistics (BJS) study looking at 30 states across the country found that recidivism is highest amongst younger individuals and nonviolent criminals.[13] Prisoners in local Louisiana prisons are therefore at a higher risk for recidivism.

According to a 2011 U.S. prison analysis by the Pew Center,[14] the most effective recidivism reduction strategy has been shown to be in-prison reform programs that employ an entry analysis of each individual's needs. These types of programs are offered in a few states across the country. However, local Louisiana prisons do not offer the same reform programs that state prisons do. In 2012 journalist Cindy Chang reported, "Most in local prisons are not even getting the basic re-entry curriculum, let alone new skills that could help them land a decent job."[15] Prisoners who leave local prisons are therefore at a higher risk of committing a second crime.

Defenders of private prisons often cite a 2001 meta-analysis of public versus private prison systems by Segal et al.[16] This study found that privately run prisons saved the state a significant amount of money and outperformed state facilities in the majority of quality standards including security, inmate safety, treatment, and health services. Segal et al. also reference a 1996 Louisiana specific study that concluded that while public systems had a higher emphasis on rehabilitation, private systems had better educational programs. Since

the release of this paper there have been no Louisiana specific comparisons between local and state prisons. Recent developments have found that due to cost cutting initiatives, local prisons have almost no substantial reform programs in place anymore,[15] and in 2007, Louisiana prisons experienced a sharp increase in prison population. Since 1996, no quality comparisons between Louisiana public and private prisons have been done in order to account for the influx of incarcerated individuals in local private prisons. The savings created by private prisons has bolstered the rate of construction of new ones. Since these private prisons must reduce costs in order to stay solvent, there is a high priority for the beds remaining occupied. This in turn makes reduction of recidivism destructive to the economic well-being of privatized jails.

One way to decrease recidivism, would be to increase employment opportunities for prisoners upon their release.[17] This can be done by guiding the integration of prisoners into the community. Currently, the Louisiana Corrections Services have GED preparatory courses, literacy education, and a vocational training that provides the inmates with technical skills.[18] When compared to the general prison population, a single cohort of released convicts who participated in Louisiana's prison education program had a 5.6% lower recidivism rate over a 5 year period.[12]

In 2011, the Economy League of Greater Philadelphia investigated how recidivism could affect the local economy. They did this by estimating the costs and benefits of having additional ex-convicts reintegrated into the community. By preventing merely one hundred ex-convicts from returning to prison, they estimated that an additional $2.7 million would be generated in combined wages and sales tax revenues over their lifetime in addition to the cost savings from the entire criminal process.[19] Although the exact numbers would not necessarily be the same, the state of Louisiana could save a significant amount of money by reducing recidivism rates across the board.

A previous study conducted in 2012 by Alvarez et al.[20] employed a mathematical model to analyze the implemented reform programs both inside and outside of the California prison system and how these programs affected recidivism rates. They found that recidivism rates were lower when educational programs were instituted outside of the prison. However, the study utilized data exclusively from the California prison system, meaning the effectiveness of in-prison educational programs for other states still remains in question. In 1997, 41.3% of the country's prison population compared to 18.4% of the general population had at most a high school education, demonstrating the stark contrast between the average prisoner and citizen.[21, 22] Although this percentage may vary across the country due to different policing standards and law practices, it is natural to assume that appropriately designed educational reform programs in prisons would effectively reduce recidivism rates.

In 2007, Seal et al.[23] analyzed how effective the Three Strikes law in California was as compared to alternative law practices, such as an infinite-strike policy. The three-strike policy sentences a criminal to life in response to their third consecutive non-violent criminal act, while an infinite-strike policy would never sentence an individual to a life term. They concluded that in higher density areas, a three-strike policy is more effective at decreasing crime, but leads to over-populated prisons. However, the infinite strike policy can still restrict crime while preventing prison overpopulation.

Our model mainly builds on the work of Alvarez et al. and Seal et al., however there have been other projects that have generated relevant results. Misra[24] used an infection model to

show how effectively population crime could be controlled in response to variable police efforts within a community. He found that maintaining a steady police force can prevent crime. McMillon et al.[25] modeled the spread of crime, imprisonment, and recidivism. They mainly focused on how longer prison sentences and higher incarceration rates affected the prevalence of crime. Interestingly, longer sentences led to greater increases in the prison population rather than decreasing the criminal population. They also found that the criminally-active population is more sensitive to changes in social welfare and education programs than it is to increased incarceration rates. This demonstrates the effectiveness of desistance, such as prisoner reentry and social programs. Misra and McMillon et al. provide us with insights into the dynamics of crime, incarceration, and sentencing in the United States. What they fail to account for is the possibility of incentivizing for-profit prisons to implement the policies that the models clearly indicate we should follow.

We seek to model the direct impact of incentivized educational programs on recidivism rates in Louisiana. A model solely monitoring first and second offenders will sufficiently answer our question. This is due to data that suggests that over 5 years a single cohort of released prisoners have a cumulative recidivism that stagnates at around 50%.[12] The objective of our model is to investigate how motivating privatized prisons to adopt educational reform programs may decrease the rate of recidivism.

In Section 2 we will formulate a model that mirrors both the incarceration dynamics of private prisons and their reaction to incentives offered by the state. Its analysis provides the conditions that optimize prison profit, seen in Section 4. Investigation into what parameters the model is most sensitive to gives insight into how important data estimates are in accurately modeling this system.

# 2 Model Development

Our ultimate objective is to find out how much incentive the state has to provide in order to reduce recidivism. In order for a prison's optimal profit strategy to reduce recidivism, a large enough incentive must be offered. This can be found by optimizing the money equation for the proportion of time spent in the reform program, and finding the subsequent conditions on the incentive being offered.

## 2.1 Assumptions

The mathematical model employs a system of difference equations, as they allow for the prison to make quarterly changes to the prisoner time allocation. This problem will be limited to a 5 year, or equivalently 20 quarter, time period. This removes the need of analyzing the possibility of second-time offenders being released and returning to prison for a third time.

We consider a constant natural death rate at every compartment of the model as well as a constant immigration of first-time offenders. In order to estimate the influx of new offenders we must assume that the number of individuals entering and leaving the local prison system are the same, allowing the influx to remain at a constant value for the 5 years that this

model runs. It is assumed that anyone leaving the released class will either reenter jail or never re-enter prison for the remainder of the model.

It is known that increased exposure to educational reform programs decreases recidivism.[12] However, our model assumes that exposure to education will have diminishing returns until the benefits become marginal. Also, these reform programs will have no effect on the average time spent incarcerated, just on recidivism rates. The amount of incentive money the prison receives is dependent on the proportion of reform time chosen for each quarter. Second time offenders undergo no educational reform program due to it being an unnecessary expenditure in the constrained time-period. Therefore the prison's revenue from second time offenders is not dependent on the amount of incentive reform being offered. Only first time offenders may participate in the reform program. To simplify our model, we assume that the first-time offenders' time is only split between labor and reform. The prison can only receive incentive money if first time offenders are participating in the reform program.

The total money the prison will make over the course of a year is the sum of the profit they will make in the first quarter, and the projected profits of the following three quarters. Each subsequent quarter after the first is valued less than the previous. We model this using a depreciation rate that is compounded quarterly. Accounting for this, the prison predetermines what proportion of reform time will optimize their total profit for each quarter.

To do this, we assume the prison has accurate information on how effective the education program is and what their decisions will do to the incarceration population dynamics. This implicitly assumes that there will be no changes in police enforcement or laws that have a large effect on recidivism or sentence length. While there is also a quantitative difference in time spent in jail for different demographics and crime types, we assume the prison population is homogeneous.

## 2.2 Incarceration Dynamics

The model divides the population of incarcerated individuals into the following classes:

- $F_t$ (first-time offenders) class: The number of first-time offenders in prison at quarter $t$. This includes a natural death proportion $\mu$. There is also a constant influx, $\Lambda$, of criminals who have committed a first offense. Any individual in this class can either move to the next class ($R_t$, released from prison class) or remain imprisoned at each time step.

- $R_t$ (released from prison) class: The number of individuals who have been released from jail and remain at risk for recidivism. There is an overall rate of individuals leaving the $R_t$ class, $\alpha$.[12] The parameter $\phi$ represents the natural proportion of individuals who become fully reformed and are never susceptible to being incarcerated a second time within our 5 year time period. The proportion of individuals for whom the educational reform program was not effective and the proportion of individuals who did not naturally reform correspond to $1 - p(q_t)$ and $1 - \phi$, respectively. We also have $p(q_t)$ which is the proportion of individuals for which the educational reform program is successful. It is a function of $q_t$, the proportion of time that inmates spent in the educational reform program. Their path in this class is split two ways at every time step $t$; they can either relapse at a portion of $\alpha$ dictated by $\big(1-p(q_t)\big)(1-\phi)$, or refrain from

committing a second crime at the portion dictated by $\left(\phi + (1-\phi)p(q_t)\right)$. The design leads to the case where if $\phi = 0$ then the rates into $S_t$ and $G_t$ are dictated exclusively by $p(q_t)$. The opposite is also true. When $p(q_t) = 0$, $\phi$ dictates the movements out of $R_t$. In addition to $\alpha$, we also have the natural death proportion $\mu$ leaving $R_t$.

- $S_t$ (second-time offenders) class: Represents the population of second-time offenders and receives an inflow of released inmates for whom the educational reform program proved to be ineffective. Because the time period being studied (5 years) is less than the average sentence length of second offenses, we consider death at rate $\mu$ to be the only way someone can leave this class.

- $G_t$ (fully reformed) class: Represents completely reformed convicts who either were positively affected by the education program or did so naturally according to the proportions $p(q_t)$ and $\phi$, respectively. In other words, they are the population of fully reformed first-time offenders.

The parameters and variables for this system are described in further detail in Tables 1 and 2.
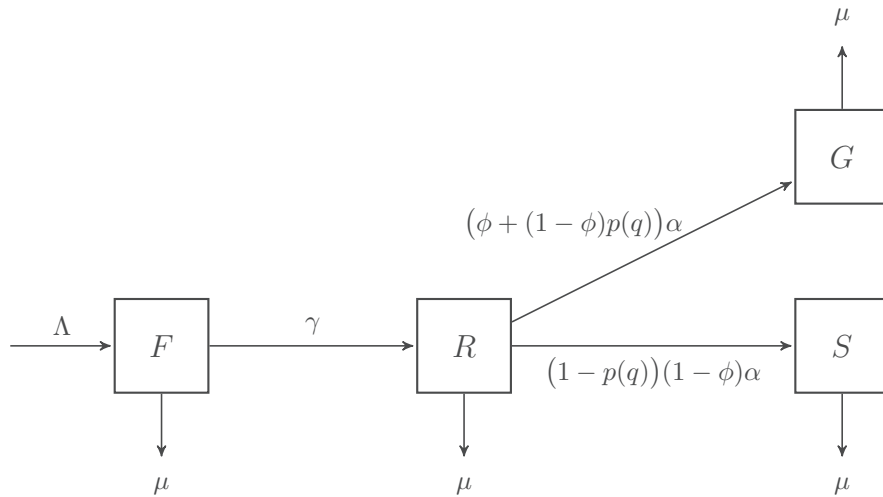


Figure 1: Model of the incarcerated population dynamics

The following system of difference equations can be used to describe the system in Figure 1.

$$F_{t+1} = \left[e^{-\gamma T}F_t + \Lambda\right]e^{-\mu T} \tag{1}$$

$$R_{t+1} = \left[\left(1 - e^{-\gamma T}\right)F_t + e^{-\alpha T}R_t\right]e^{-\mu T} \tag{2}$$

$$S_{t+1} = \left[\left(1 - p(q_t)\right)\left(1 - \phi\right)\left(1 - e^{-\alpha T}\right)R_t + S_t\right]e^{-\mu T} \tag{3}$$

$$G_{t+1} = \left[\left(\phi + \left(1 - \phi\right)p(q_t)\right)\left(1 - e^{-\alpha T}\right)R_t + G_t\right]e^{-\mu T} \tag{4}$$

Table 1: Description of the variables for the incarceration dynamics system as described in Figure 1.

| Variable | Description | Unit |
|---|---|---|
| $F_t$ | Population of first-time offenders in the correctional facility | People |
| $R_t$ | Population of released first-time offenders | People |
| $S_t$ | Population of second-time offenders in the correctional facility | People |
| $G_t$ | Population of reformed first-time offenders | People |
| $q_t$ | Proportion of an offender's time spent in a reform program | - |

Table 2: Description of the parameters for the incarceration dynamics system as described in Figure 1.

| Parameters | Description | Unit |
|---|---|---|
| $p(q_t)$ | Probability of success of prison reform program | - |
| $\gamma$ | Rate of release from prison | $\frac{1}{\text{Time}}$ |
| $\alpha$ | Rate at which individuals leave the released class | $\frac{1}{\text{Time}}$ |
| $\phi$ | Natural proportion of individuals who become law abiding citizens | - |
| $\mu$ | Natural death rate | $\frac{1}{\text{Time}}$ |
| $\Lambda$ | Number of first-time offenders entering the system | People |

## 2.3   Money Dynamics

The money dynamics are influenced by the success of the reform program, the profit made by the prison, and the future profit of the prison, accounting for depreciation.

Since reform programs are not guaranteed to be 100% successful, there is a maximum amount of reduction, $p_{max}$. This parameter can be thought of as the "carrying capacity" of the $p(q_t)$ function. We choose the percent success of the program to follow a pseudo-logarithmic growth modeled by the Verhulst function. Verhulst functions have the desired

carrying capacity property and have the advantage of closely modeling the Section 2.1 idea of diminishing returns. We incorporate a scaling factor, $n + 1$, to force the condition that $p(q_t = 1) = p_{max}$ for the Verhulst function. This means that even if the prison allocated all of the prisoners' time towards the educational reform program the program would still only observe a rate of success bounded by $p_{max}$. Equation (5) represents the probability of success of the reform program.

$$p(q_t) = p_{max}(n+1)\frac{q_t}{q_t + n}, \qquad q_t \in [0, 1] \tag{5}$$

First-time offenders are subjected to a reform program in order to prevent them from returning to prison once they are released. The parameter $d$ is the currently administered per quarter payment the prison receives from the state per prisoner, $f$ is the per quarter fixed cost of each prisoner to the prison, $L$ is the amount of money the prison earns per prisoner per quarter off of inmate labor, and $r$ is the cost of the reform program per prisoner per quarter. Since second time offenders do not participate in the reform program, the revenue the prison generates from them is only affected by $d$, $f$, and $L$. Equation (6) accounts for the profit per quarter the prison generates for first and second time offenders.

$$\text{Profit}_t = \left[(d - f + L)S_t + \left(L(1 - q_t) + kp(q_t) + d - f - rq_t\right)F_t\right]T \tag{6}$$

The profit per quarter equation (6) is either explicitly or implicitly a function of $q_t$, $S_t$, and $F_t$; these values are based on the current time step and all those prior. We project future profits per quarter for the upcoming year, optimize the function with respect to $q_t$, where $t$ is measured in quarters, and repeat the cycle once we reach the next year. Future quarters within a year are valued less at the depreciation rate $\delta$. Since there are four quarters in a year, we can look at most three quarters into the future ($y = 3$) or at least zero quarters into the future ($y = 0$). Equation (7) is the current and the future value of the profit with depreciation.

The term in the profit equation associated with first-time offenders is affected by the proportion of time that they spend participating in the reform program. At the beginning of every year the prison determines what $q_{t+y}$ will optimize profit. Thus $q_{t+y}$ is calculated for the next four quarters at the beginning of each year so $q_t, q_{t+1}, q_{t+2}$, and $q_{t+3}$ are found such that they optimize $\text{Money}_t$.

$$\text{Money}_t = \sum_{i=t}^{t+y} \delta^{i-t}\text{Profit}_i(\{q_j\}_{j=t}^i, \{S_j\}_{j=t}^i, \{F_j\}_{j=t}^i) . \tag{7}$$

Table 3: Description of the parameters within the profit maximization system.

| Parameters | Description | Unit |
|---|---|---|
| $d$ | State allocated money per inmate per quarter | $\frac{\text{Dollars}}{\text{Time} \times \text{People}}$ |
| $f$ | Fixed cost to prison per inmate per quarter | $\frac{\text{Dollars}}{\text{Time} \times \text{People}}$ |
| $L$ | Labor revenue per inmate per quarter | $\frac{\text{Dollars}}{\text{Time} \times \text{People}}$ |
| $k$ | Incentive to host reform program per inmate per quarter | $\frac{\text{Dollars}}{\text{Time} \times \text{People}}$ |
| $r$ | Cost of reform program per inmate per quarter | $\frac{\text{Dollars}}{\text{Time} \times \text{People}}$ |
| $T$ | Length of time step | Time |
| $p_{max}$ | Maximum proportion effect of reform program on recidivism | - |
| $n$ | Half-saturation level for $p(q_t)$ function | - |
| $\delta$ | Depreciating value of future assets per quarter | - |
| $y$ | The number of time steps beyond the current one | - |

The system couples profit optimization with an incarceration process, as shown in Figure 2. The prison decides what proportion of the first-time offender's time $q_t$ will be spent in a reform program in order to maximize the profit earned from the state's offered incentive $k$. After finding the optimal proportion of time spent in a recidivism program the incarceration process progresses in time. This is very similar to the way that human decision processes may influence the epidemiological landscape over which a disease evolves, which in turn adjusts the costs and benefits used in future human decisions.[26–28]
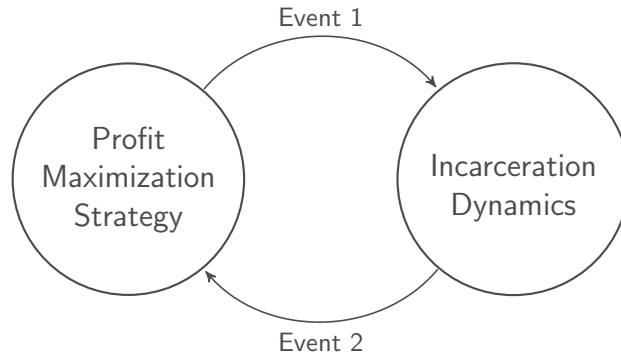


Figure 2: Flowchart of the adaptive system.

# 3  Parameter Estimation from Literature

Table 4: Table of all the parameters for the incarceration dynamics system as described in Figure 1 and the money dynamics system, all with citations.

| Parameters | Estimation |
|---|---|
| $F_0$ | 1000 prisoners[12] |
| $T$ | 1 Quarter (3 months) |
| $\gamma$ | 0.06931 $\frac{1}{\text{Quarter}}$ [12] |
| $\alpha$ | 0.115 $\frac{1}{\text{Quarter}}$ [12] |
| $\Lambda$ | 50 People[12] |
| $\mu$ | 0.00028 $\frac{1}{\text{Quarter}}$ [29] |
| $\phi$ | 0.52 [12] |
| $p_{max}$ | 0.1, 0.2, or 0.3 [12, 30] |
| $n$ | 1 |
| $\delta$ | 0.9914 [31] |
| $d$ | \$2219.49 $\frac{1}{\text{Prisoner} \times \text{Quarter}}$ [9] |
| $r$ | \$392.21 $\frac{1}{\text{Prisoner} \times \text{Quarter}}$ [32] |
| $f$ | \$2037.49 $\frac{1}{\text{Prisoner} \times \text{Quarter}}$ [9] |
| $L$ | \$182 $\frac{1}{\text{Prisoner} \times \text{Quarter}}$ [33] |
| $y$ | 0-3 |

The first set of parameter estimations help us employ an accurate model of incarceration dynamics. Due to our model being based on the decisions of a theoretical prison, we decided to create an initial first-offender class that was a fraction of the 20,000 prisoners in the Louisiana local prison system, i.e. $F_0 \approx 1000$. Since most businesses make budget decisions every quarter, it follows that the time interval, $T$, should be 3 months, or one quarter of a year. This led to the necessity of scaling our parameter estimations into quarterly quantities. To find the rate at which first time offenders are released we used recent data from Louisiana local prisons stating that the average time served is 2.5 years, or 10 quarters. Assuming a symmetric distribution of incarceration time around the mean, half of the prisoners will have left the prison after 2.5 years. Thus, $\frac{F_{10}}{F_0} = 0.5$. It follows that $\gamma = \frac{-\ln 0.50}{10} \approx 0.06931 \frac{1}{\text{Quarter}}$. Individuals will then move from the released class to either reform or back to prison. The rate at which this happens is based on data that follows cohorts and records recidivism over a 5 year period. We calculated this value, $\alpha$, by assuming that after 5 years, or 20 quarters, $\frac{R_{20}}{R_0} = 0.1$. This assumes that 90% of individuals in the released class will leave by the 5$^{\text{th}}$ year. It follows that $\alpha = \frac{-\ln 0.10}{20} \approx 0.115 \frac{1}{\text{Quarter}}$. The proportion of released inmates who enter the reformed class or return to prison is always based on $\phi$. We assumed this proportion of individuals per quarter entering the reformed class from the released class would be equivalent every quarter when no reform program is in place. Since after 5 years the cumulative recidivism in Louisiana for a cohort of released prisoners is around 48% we will be approximating $\phi = 1 - 0.48 \approx 0.52$. To find the number of new first time offenders entering the theoretical prison per quarter we use data of the total number of criminals entering local prisons per year. Since our initial prison size contains 1,000 individuals at

$t = 0$, it contains $\frac{1}{20}$ of the entire local prison population. Every year the number of released and admitted prisoners from local Louisiana prisons is approximately 8,000 individuals each. Due to these values being approximately the same we can use recidivism data to assume that around 50% of the newly admitted inmates are entering due to recidivism. Thus, around 4,000 of the newly admitted inmates will be first-time offenders. Assuming that this parish receives a proportional number of new offenders compared to the rest of the local prisons, $\Lambda = \frac{\text{Init. Prison Pop. Size } (F_0)}{\text{Total Local Prison Pop.}} \times \frac{\text{\# First Time Offenders}}{\text{Quarter}} = \frac{1,000}{20,000} \times \frac{4,000}{\text{Year}} \times \frac{\text{Year}}{4 \text{ Quarters}} \approx 50$ People. Due to the assumption that the prison population is homogeneous, we estimated the death rate for the entire system by first finding that the average age in local Louisiana prisons is 33.9 years of age.[12] Using CDC data we find that individuals of the ages 30-34 have a 114 per 100,000 people death rate. This means that around 0.11% of the population dying every year. Thus the death rate per quarter, $\mu = \frac{-\ln(1-.0011)}{4} \approx 0.00028 \frac{1}{\text{Quarter}}$.

In order to accurately model how prison profit changes we estimate the parameters found in the profit equation. The current education system in the local Louisiana prison system has successfully reduced recidivism by around 6% over a 5 year period. Rounding up, we get that $p_{max} \approx 0.1$. However, some education programs have been found to decrease recidivism by 30%, so $p_{max}$ can theoretically reach this level if the program is successful. Due to data demonstrating that values of 0.1 and 0.3 are realistic we also analyze the intermediate value, when $p_{max} = 0.2$. From the probability that the reform program is successful, $p(q_t)$, we can derive that $\frac{\partial p(q_t)}{\partial q_t} = p_{max}(n+1)\frac{n}{(q_t+n)^2}$. So when $q_t = 0$ we get $\frac{p_{max}(n+1)}{n}$. Thus the initial slope for this equation is dependent on $p_{max}$ and $n$. There is no data on this slope and since we already estimated $p_{max}$, we will let $n = 1$. This will lead to the initial slope being $2 \times p_{max}$. In order to fully comprehend how a prison will maximize profit we can use current interest rates. To find the depreciating value of future assets we can use the 2015 interest values of 3.5 % per annum to estimate that $\delta = \left(\frac{\text{Value of Money Earned A Year From Now}}{\text{Value of Money Earned Now, After A Year}}\right)^{\frac{1}{4}} = \left(\frac{1}{1.035}\right)^{\frac{1}{4}} = (0.9662)^{\frac{1}{4}} \approx 0.9914$. The current state-allocated money paid to local prisons is set at \$24.39 per day per prisoner. We must convert this to a per quarter value so we do $d = \$24.39 \frac{1}{\text{Prisoner} \times \text{Day}} \times \frac{91 \text{ Days}}{\text{Quarter}} = \$2219.49 \frac{1}{\text{Prisoner} \times \text{Quarter}}$. In order to determine how much educational reform programs cost we used a recent analysis by the RAND Corporation that estimates it costs \$3.84 to \$4.78 per day per prisoner. We then take the average of these two values to arrive at \$ 4.31 per day per prisoner. To convert from days to quarters we do $r = \$4.31 \frac{1}{\text{Prisoner} \times \text{Day}} \times \frac{91 \text{ Days}}{\text{Quarter}} = \$392.21 \frac{1}{\text{Prisoner} \times \text{Quarter}}$. Since $f$ is how much the average convict costs this particular prison, $d - f$ will be the profit. The smaller $f$ is the more effective the prison will be at lowering these costs. For later sensitivity analysis we will assume that the prisons will be earning \$2 per day, so converting to quarters we get $f = \$22.39 \frac{1}{\text{Prisoner} \times \text{Day}} \times \frac{91 \text{ Days}}{\text{Quarter}} = \$2037.49 \frac{1}{\text{Prisoner} \times \text{Quarter}}$. As this value decreases, the prison makes more money off each prisoner. Another value that does this is the amount of money the prison makes per day off of the labor of each prisoner. This is dependent on how many of the prisoners are participating in work programs. Using revenue data from the largest prison corporation in the country (CCA), we find the average income earned from each of their 91,000 prisoners a day to be $\frac{\$40,522,000}{91,000} \times \frac{\text{quarter}}{91 \text{ days}} \approx \$4.89 \frac{1}{\text{Prisoner} \times \text{Day}}$. Since this income comes from various sources outside of labor, we will use this as a cap on the maximum amount of money the prison can earn from each prisoner through labor. We

use a conservative estimate of an average of \$2 per day per prisoner. Applying the proper conversion we get $L = \$2 \frac{1}{\text{Prisoner} \times \text{Day}} \times \frac{91 \text{ Days}}{\text{Quarter}} = \$182 \frac{1}{\text{Prisoner} \times \text{Quarter}}$. The parameter $y$ is the number of steps beyond the (i.e. $y \in [0,3]$ since there are only four quarters in a year).

# 4  Results

We optimize the Money equation (7) with respect to the time spent in a reform program, $q_t$. Each term of the Money equation is the Profit equation (6), scaled by its corresponding depreciation factor $\delta$. Before we can optimize Money with respect to $q_t$ we must find the derivative of Profit, according to

$$\frac{\partial \text{Profit}_t}{\partial q_t} = \left[ \left( L + d - f \right) \frac{\partial S_t}{\partial q_t} + \left( k \frac{\partial p(q_t)}{\partial q_t} - r - L \right) F_t \right] T.$$

To obtain a full expression for Profit, we need explicit formulas for $F_t$ and $S_t$. We will derive the equations in the following sections.

## 4.1  Incarceration Dynamics

Assuming all $q_t$ values are changing at each time step we can solve for the difference equations. First recall the set of difference equations used to describe the incarceration dynamics and profit maximization in Figure 2.

We replace the coefficients in our equations according to

$$m = e^{-\mu T},$$
$$a = e^{-\gamma T},$$
$$b = e^{-\alpha T},$$
$$c(q_t) = \left(1 - p(q_t)\right)\left(1 - \phi\right),$$
$$h(q_t) = \left(\phi + \left(1 - \phi\right)p(q_t)\right).$$

The parameter $m$ is the proportion of people who survive natural death after each time step, $a$ is the proportion of people who remain in $F_t$ after each time step, and $b$ is the proportion of people who remain in $R_t$ at each time step. $c(q_t)$ is the proportion of people, who upon leaving $R_t$, will go to $S_t$, and $h(q_t)$ is the proportion of people leaving $R_t$ who will enter $G_t$.

Substituting these into the incarceration dynamics equations, we can simplify them to:

$$\frac{1}{m} \begin{bmatrix} F_{t+1} \\ R_{t+1} \\ S_{t+1} \\ G_{t+1} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ (1-a) & b & 0 & 0 \\ 0 & c(q_t)(1-b) & 1 & 0 \\ 0 & h(q_t)(1-b) & 0 & 1 \end{bmatrix} \begin{bmatrix} F_t \\ R_t \\ S_t \\ G_t \end{bmatrix} + \begin{bmatrix} \Lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The Profit equation is independent of $G_{t+1}$, therefore we will not need it nor $h(q)$.

Solving for the difference equations, we find

$$F_t = (am)^t F_0 + \Lambda m \frac{1 - (am)^t}{1 - am},$$

$$R_t = (bm)^t R_0 + (1 - a) \sum_{i=0}^{t-1} b^{t-1-i} m^{t-i} F_i,$$

$$= (bm)^t R_0 + (1 - a) \sum_{i=0}^{t-1} b^{t-1-i} m^{t-i} \left( (am)^i F_0 + \Lambda m \frac{1 - (am)^i}{1 - am} \right),$$

$$S_t = m^t S_0 + (1 - b) \sum_{i=0}^{t-1} c(q_i) m^{t-i} R_i,$$

$$= m^t S_0 + (1 - b) \sum_{i=0}^{t-1} c(q_i) m^{t-i} \left( (bm)^i R_0 + (1 - a) \sum_{k=0}^{i-1} b^{i-1-k} m^{i-k} \left( (am)^k F_0 + \Lambda m \frac{1 - (am)^k}{1 - am} \right) \right).$$

The steps for the analysis in this section can be found in the Appendix Section 7.1.

## 4.2  Optimizing Money

At each time step of this model, we find the optimal $q_t$. In this section, we explain how to optimize this $q_t$. First we consider the $S_{t+y}$ equation, (given $y$ is a number of time steps after the $t^{\text{th}}$ step) with a changing $q_t$ to find the most general solution.

$$S_{t+y} = m^{t+y} S_0 + (1 - b) \sum_{i=0}^{(t+y)-1} c(q_i) m^{(t+y)-i} R_i.$$

Note that $S_{t+y}$ is dependent on all $q_i$ before $q_{t+y}$. Therefore, in the Money equation (7), all terms are explicitly or implicitly dependent on $q_t$. To optimize the Money equation with respect to $q_t$, we first expand it as

$$
\begin{aligned}
\text{Money}_t = {}& \text{Profit}_t(q_t, S_t, F_t) + \delta \text{Profit}_{t+1}(q_t, q_{t+1}, S_t, S_{t+1}, F_t, F_{t+1}) \\
& + \delta^2 \text{Profit}_{t+2}(q_t, q_{t+1}, q_{t+2}, S_t, S_{t+1}, S_{t+2}, F_t, F_{t+1}, F_{t+2}) \\
& + \delta^3 \text{Profit}_{t+3}(q_t, q_{t+1}, q_{t+2}, q_{t+3}, S_t, S_{t+1}, S_{t+2}, S_{t+3}, F_t, F_{t+1}, F_{t+2}, F_{t+3}).
\end{aligned}
$$

Next take the partial derivative of Money with respect to $q_t$ and find

$$
\begin{aligned}
\frac{\partial \text{Money}_t}{\partial q_t} = {}& \left[ (-L - r + k \frac{\partial p(q_t)}{\partial q_t}) F_t + \delta(d - f + L) \frac{\partial S_{t+1}(q_t, q_{t+1})}{\partial q_t} \right. \\
& \left. + \delta^2(d - f + L) \frac{\partial S_{t+2}(q_t, q_{t+1}, q_{t+2})}{\partial q_t} + \delta^3(d - f + L) \frac{\partial S_{t+3}(q_t, q_{t+1}, q_{t+2}, q_{t+3})}{\partial q_t} \right] T,
\end{aligned}
$$

where for $y > 0$

$$\frac{\partial S_{t+y}}{\partial q_t} = (1-b)\frac{\partial c(q_t)}{\partial q_t}m^y R_t,$$

$$\frac{\partial c(q_t)}{\partial q_t} = -\frac{\partial p(q_t)}{\partial q_t}(1-\phi),$$

$$= -\left[p_{max}(n+1)\frac{n}{(q_t+n)^2}\right](1-\phi),$$

Substituting $\frac{\partial c(q_t)}{\partial q_t}$ into the $\frac{\partial S_{t+y}}{\partial q_t}$ equation, we can simplify to find

$$\frac{\partial S_{t+y}}{\partial q_t} = -(1-b)(1-\phi)\left[p_{max}(n+1)\frac{n}{(q_t+n)^2}\right]m^y R_t.$$

Solving $\frac{\partial \text{Money}_t}{\partial q_t} = 0$ for $q_t^*$, the optimal value of $q_t$, when $y > 0$ yields

$$q_t^* = -n \pm \sqrt{n(n+1)p_{max}\left(\frac{kF_t - (d-f+L)(1-b)(1-\phi)R_t\sum_{i=1}^y(\delta m)^i}{F_t(L+r)}\right)}.$$

We find each quarter's $q_t^*$, accounting for the depreciation from the remaining quarters in the year. To do this, we vary $y$ between 0 and 3, depending on the quarter at which we are situated within the year. Since only non-negative values of $q_t$ are of interest, we only consider the larger root. $q_t^*$ is non-negative only when

$$\frac{(n+1)p_{max}}{n} = \frac{\partial p(q_t)}{\partial q_t}\bigg|_{q_t=0} \geq \frac{(L+r)F_t}{kF_t - (d-f+L)(1-b)(1-\phi)R_t\sum_{i=1}^y(\delta m)^i},$$

and real only when

$$kF_t \geq (d-f+L)(1-b)(1-\phi)R_t\sum_{i=1}^y(\delta m)^i.$$

$\frac{\partial p(q_t)}{\partial q_t}$ when $q_t = 0$ can be interpreted as the change in efficiency of the reform program. The right hand side of the non-negativity condition is the maximum projected cost of the reform program compared to the total gain from having the reform program (the maximum projected gain from the incentive minus the projected future loss from potential second time offenders). Therefore, for $q_t^*$ to be non-negative, the change in efficiency of the reform program at $q_t = 0$ must be greater than the relative expected cost of the reform program compared to the expected revenue gain from the reform program. The steps for finding this condition can be found in Appendix Section 7.4.

With these conditions met, we can solve the equation for $q_t^*$ with respect to the incentive, $k$, and find the boundary conditions $q_t^* = 0$ (no time in the reform program) or 1 (all time spent in the reform program). The general boundaries on $k$ when $y > 0$ are

$q_t^* = 0$ for $k \leq k_{min}$, where

$$k_{min} = \frac{n}{(n+1)p_{max}}(L+r) + \frac{(d-f+L)(1-b)(1-\phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t}, \quad (8)$$

whereas $q_t^* = 1$ for $k \geq k_{max}$, where

$$k_{max} = \frac{n+1}{(n)p_{max}}(L+r) + \frac{(d-f+L)(1-b)(1-\phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t}. \quad (9)$$

For $k_{min}$, the first term is the same as the left hand side of the condition of existence of $q_t^*$ described above. Therefore the first term can be summarized to mean the projected maximum profit loss per percent reduction in recidivism per quarter, per prisoner, when $q_t^* = 0$. The second term of $k_{min}$ is the projected profit depreciation from future second time offenders without reform program per first time offender per quarter per proportion of time spent in the reform.

For $k_{max}$, $\frac{n+1}{(n)p_{max}}$ can be interpreted as the increase in efficiency of the reform program when $q_t^* = 1$. Therefore we can interpret this first term as the projected maximum profit loss per percent reduction in recidivism per quarter per prisoner when $q_t^* = 1$. The second term of $k_{max}$ is the same as that of $k_{min}$. It is the projected profit depreciation from future second time offenders without a reform program per first time offender per quarter per proportion of time spent in the reform program.

When $k$ has a value between $k_{min}$ and $k_{max}$, it is possible to find a $q_t^*$ such that $q_t^* \in (0,1)$. When $k$ is less than Condition (8), $q_t^*$ is always zero. When $k$ is greater than Condition (9), $q_t^*$ is always one. When $y = 0$, we are not concerned with the future, thus $\delta = 1$. This yields the simplified conditions

$$q_t^* = -n \pm \sqrt{\frac{n(n+1)p_{max}k}{(L+r)}},$$

for $q_t^* = 0$

$$k_{min} = \frac{n}{(n+1)p_{max}}(L+r),$$

and for $q_t^* = 1$

$$k_{max} = \frac{(n+1)}{np_{max}}(L+r),$$

used only for analysis on the 4$^{\text{th}}$ quarter of each year.

### 4.2.1 Equilibrium of the Incarceration Dynamics

To find the bounds on $k_{min}$ and $k_{max}$ as time approaches infinity, we need to know the equilibrium values for $F$ and $S$. The equilibrium for the incarceration equations is the

saturation value for each class of the prison system we are modeling. For example $F_\infty$ is the component the tells us the equilibrium number of first-time offenders in this prison system as time approaches infinity. The equilibrium is

$$F_\infty = \frac{\Lambda m}{(1 - am)},$$

$$R_\infty = \frac{(1 - a)m F_\infty}{(1 - bm)},$$

$$S_\infty = \frac{c(q_\infty)(1 - b)m R_\infty}{(1 - m)},$$

$$G_\infty = \frac{h(q_\infty)(1 - b)m R_\infty}{(1 - m)}.$$

The steps for the analysis in this section can be found in the Appendix Section 7.2. From these equations, we can see that the equilibrium of each class is dependent on the rate of inflow over the rate of outflow from the class, moderated by $m$, the survival rate.

## 4.3   Numerical Results with Parameter Estimations

In this section, we summarize the numerical results found after using the estimated parameter values from Section 3. Using various values for the reform effectiveness parameter, $p_{max}$, its relationship with state incentives becomes apparent. For each of these $p_{max}$ values we calculated the upper and lower bound for the incentive value provided by the state that will ensure the prison allocates all or none of a prisoner's time in a reform program. These values are represented as $k_{max}$ and $k_{min}$ respectively, as demonstrated in Table 5. We computed these values using methods found in Appendix 7.3. For the following analysis, we define the percent reduction in recidivism as

$$\left(1 - \frac{S_{20}(q_{1,\dots,20} = \bar{q}_t)}{S_{20}(q_{1,\dots,20} = 0)}\right) 100\% = X\%,$$

where $\bar{q}_t$ is the mean $q_t$ value for a specific $k$. This is equivalent to taking the incentive value that creates a second-offense class that is 90% of the size of this same class when $k \leq k_{min}$. We performed this analysis using various reform program effectivenesses. Figure 3 demonstrates an example of the percent recidivism after 5 years versus $k$ when $p_{max} = 0.3$. For this study, we will be choosing $X = 10\%$. From here the type of incentive that creates a relative 10% reduction in recidivism was found for a given $p_{max}$. This is summarized in Table 5. Thus, $k_{10}$ is the incentive the state needs to offer per quarter to create a 10% reduction in recidivism, $\bar{q}_t$ is the average allocation of time associated with this reduction and $\bar{k}_d$ is how much incentive this translates to per day for the prison. Using the average appropriation of time per prison, $\bar{q}_t$, that creates a 10% reduction in relative recidivism we can determine what is the necessary average additional funding per prisoner per day. This is done by finding

$$\bar{k}_d = k \times p(\bar{q}_t)\left(\frac{1 \text{ quarter}}{91 \text{ Days}}\right).$$

|                      | $p_{max}$ | | |
| :---: | :---: | :---: | :---: |
| Associated variables | 0.1 | 0.2 | 0.3 |
| $k_{min}$ | $2868.18 | $1469.96 | $991.40 |
| $k_{max}$ | $11481.33 | $5776.494 | $3862.45 |
| $k_{10}$ | $\geq$ $11481.33 | $2566.00 | $1391.50 |
| $\bar{k}_d$ | $12.62 | $2.82 | $1.53 |
| $\bar{q}_t$ | 1 | 0.3341 | 0.2011 |
| 5 Year Profit | $49 Million | $22.8 Million | $20.9 Million |

Table 5: The $k_{min}$ and $k_{max}$ are the lower and upper boundary, between which the prison alters $q_t$. From this, the $k_{10}$, $\bar{q}_t$, and average additional per prisoner per diem provided by the state, $\bar{k}_d$, are calculated. These are the respective values that reduce relative recidivism by 10%. The total money earned for each $p_{max}$ was calculated by summing the money earned from each quarter over 5 years. The baseline total money earned over 5 years by the prison is $20 Million.
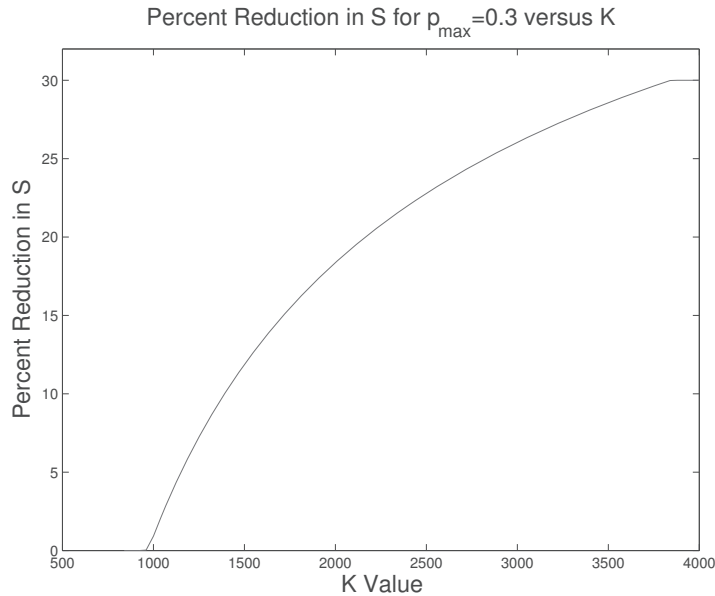


Figure 3: Percent reduction in $S_{20}$, or in other words, the percent reduction in $S$ after 5 years, versus $k$, the incentive value

Figures 4, 5, and 6 show the expected population dynamics when we reduce recidivism by 10% in each of the three cases shown in Table 5.
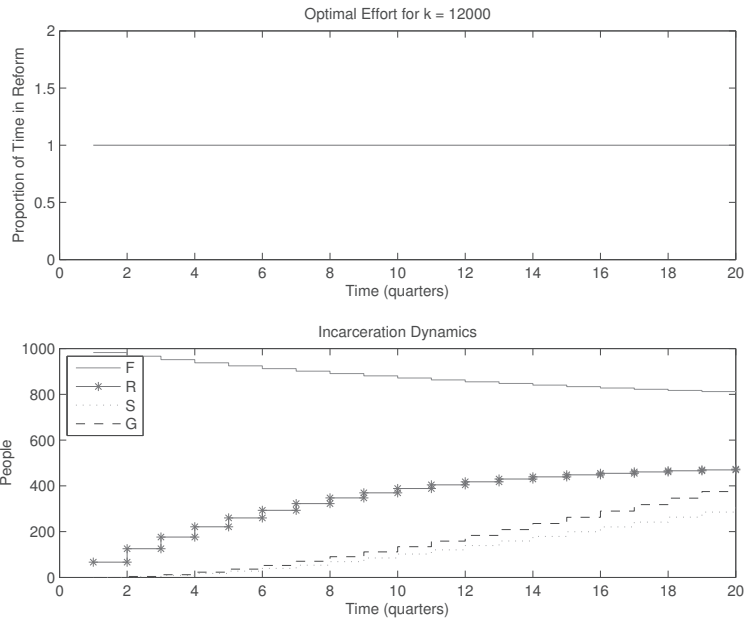
Figure 4: Optimal $q_t$ and expected population when $p_{max} = 0.1$
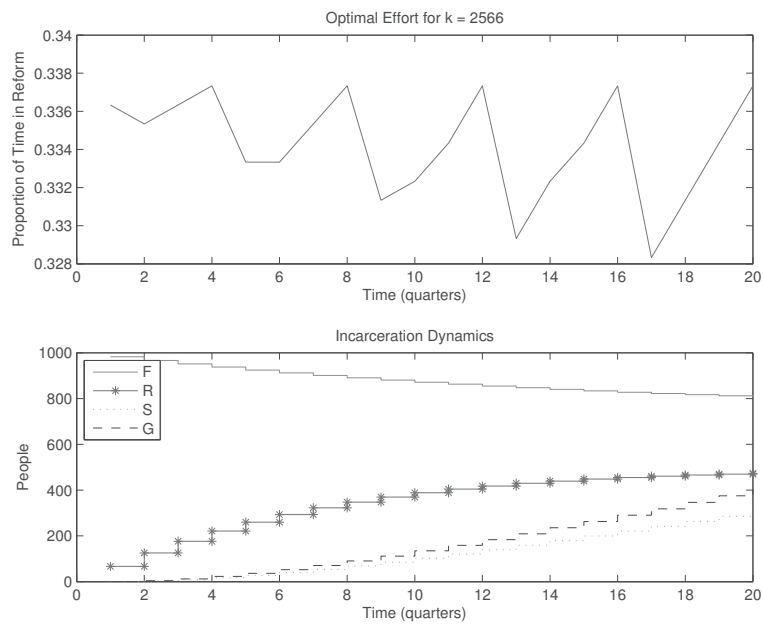


Figure 5: Optimal $q_t$ and expected population when $p_{max} = 0.2$
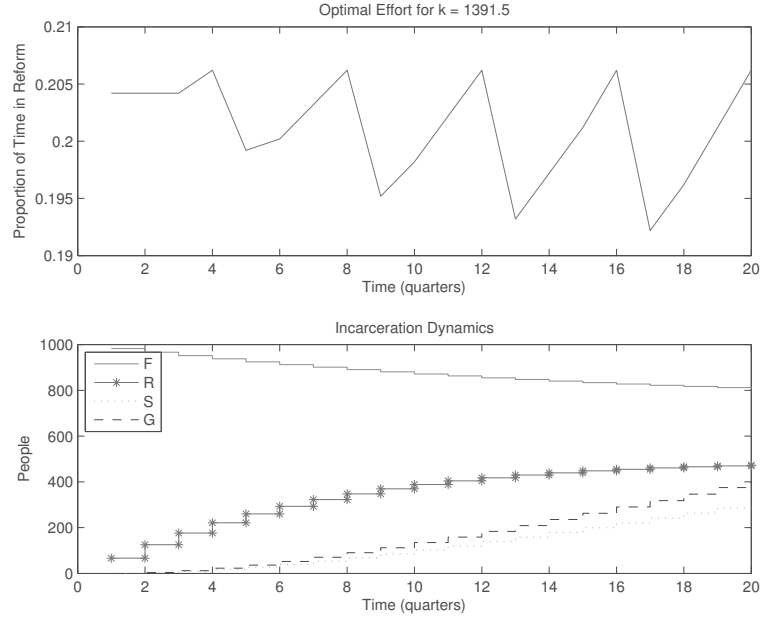
Figure 6: Optimal $q_t$ and expected population when $p_{max} = 0.3$

## 4.4 Local Sensitivity Analysis

Holding all other parameters constant, local sensitivity is the proportional change of a function with respect to one of its influencing parameters. We perform the local analyses by increasing the parameter by 1% and analyze the results. Sensitivity indexes are calculated according to

$$\text{Sensitivity}(Y_t, x) = \frac{\partial Y_t}{\partial x} \frac{x}{Y_t} \ ,$$

where $x$ is the parameter being altered and $Y_t$ is the state variable.

### 4.4.1 Profit Equation Analysis

The sensitivity indexes for all parameters and state variables in the Profit Equation can be found in Appendix 7.5

As an example, the sensitivity indexes for $\text{Profit}_t$ at the beginning of years 1, 2, 3, and 4 are summarized in Figure 7 for $p_{max} = 0.3$. Note that all the $q_t$ values used in sensitivity indexes are from the 1st quarter of every year.
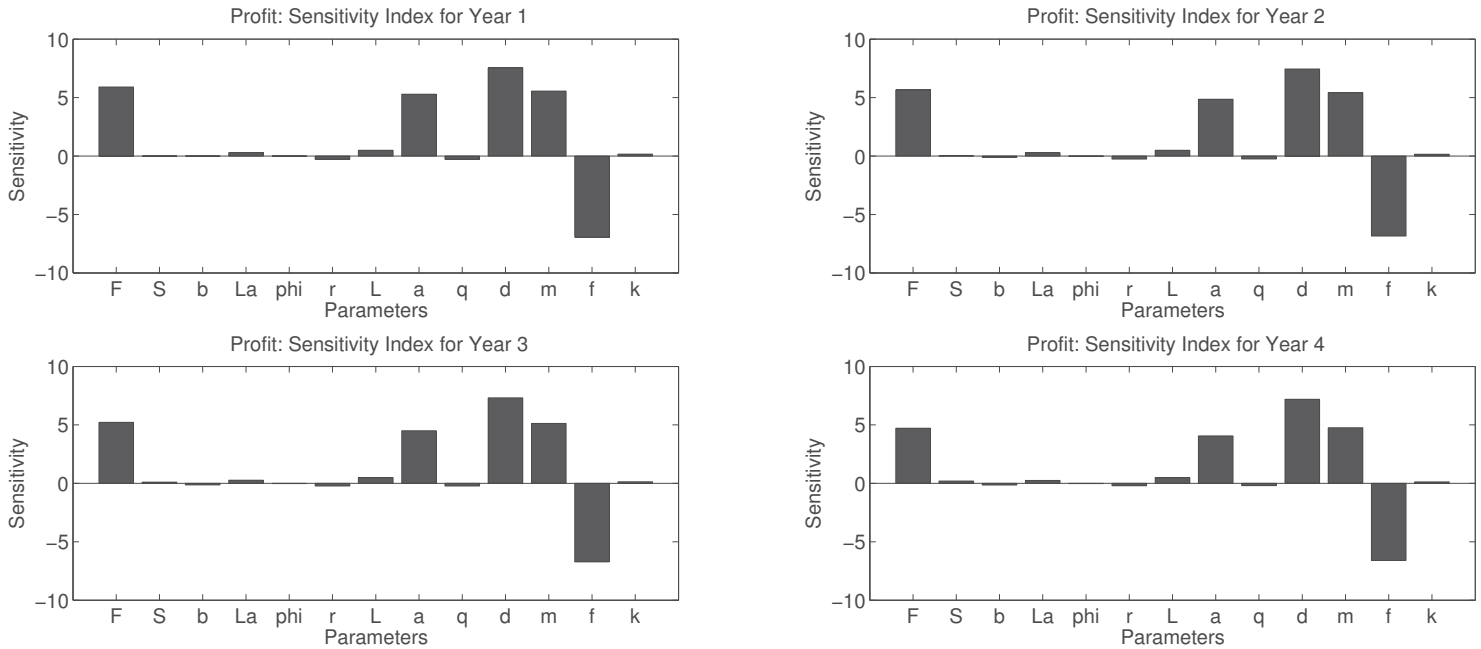
Figure 7: Sensitivity indices of Profit when $p_{max} = 0.3$

Each value shown in Figure 7 can be interpreted as the proportional change in profit as a result of a one percent change in the value of the parameter in question.

The graphs show that the local sensitivity of the Profit Equation is extremely responsive to changes in the fixed cost per prisoner, $f$, and the amount received per prisoner per day per quarter, $d$. If a prison can decrease their fixed costs they can greatly increase their profits. It is also sensitive to $m$, the survival probability, and $a$, retention probability of first time offenders.

### 4.4.2 Incarceration Difference Equations Analysis

The sensitivity analysis was performed on all parameters including $a, b, c(q_t), \Lambda, m,$ and $\phi$. This can be found in Appendix 7.6

Using the parameter estimations, we can find the sensitivity indexes with respect to $F_t$, $R_t$, and $S_t$. Figure 18 shows an example of the results for $p_{max} = 0.3$. We only show local and global stability tables in terms of $p_{max} = 0.3$ because the general shape of these histograms are uniform across the analyzed $p_{max}$ values. These other sensitivity indexes for $p_{max} = 0.2$ or 0.1 can be found in Appendix 7.7. Note that all the $q_t$ values used in sensitivity indexes are from the $1^{st}$ quarter of every year.
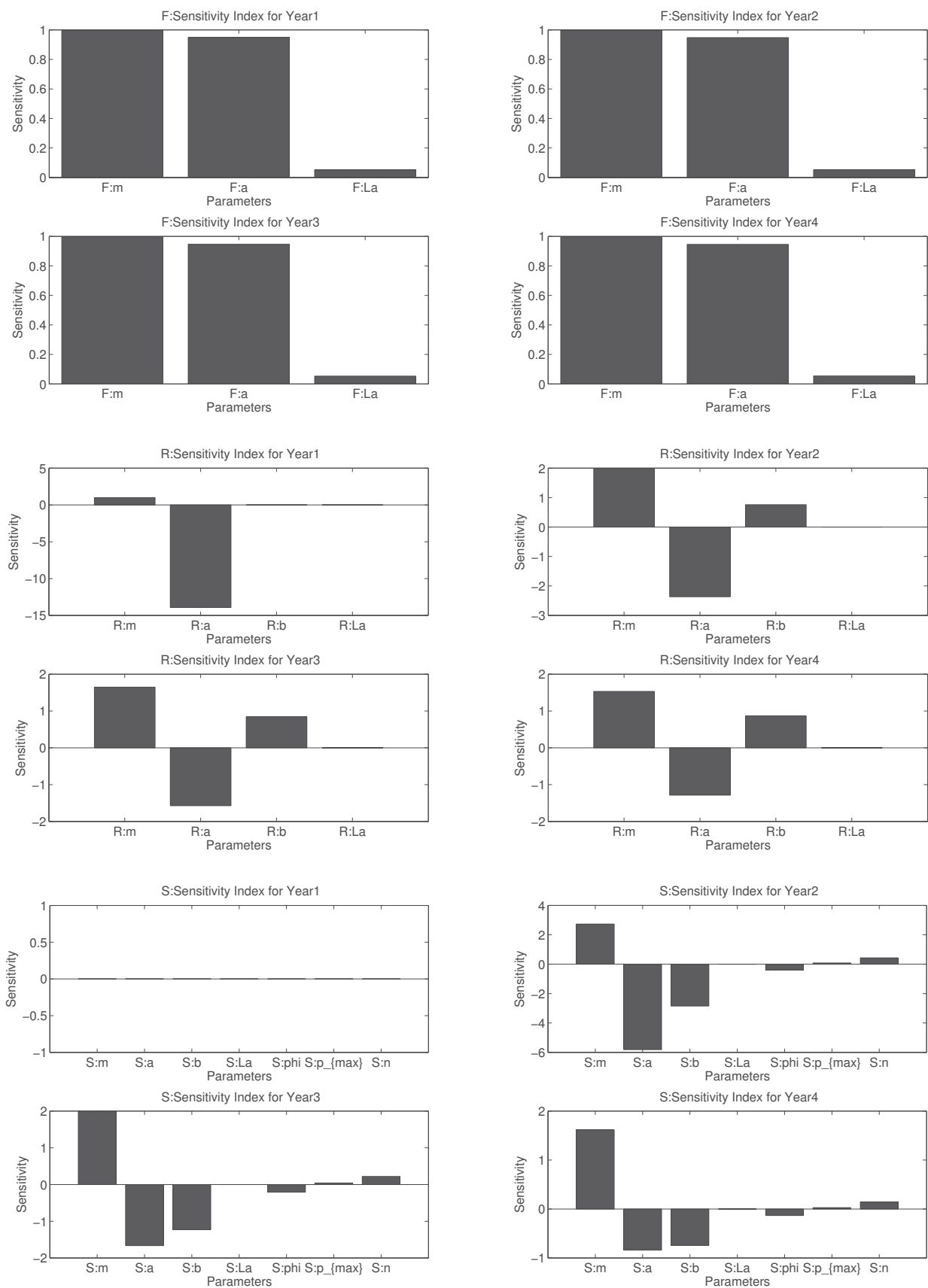
Figure 8: Sensitivity indexes of $F$, $R$, and $S$ respectively when $p_{max} = 0.3$
Note scales are different to show the difference in relative magnitude in each separate year.

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Our analyses on the local sensitivity, pictured in Figure 18, feature $F$, $R$ and $S$, which are all directly related to $m$. This follows our intuition since $m$ is the proportion of individuals who survive every time step, so an increase in its value would lead to an increase in the number of people in each class at each time step relative to a smaller $m$. The parameter $a$, the proportion of people who stay in $F$, is directly related to $F$, and inversely related to the two other classes. On the other hand, $b$, the proportion of individuals who remain in $R$, has no effect on $F$, but is directly related to $R$ and inversely with respect to $S$. Finally, $\phi$ only affects $S$, and does so negatively, since increasing $\phi$ increases the proportion of the flow out of $R$ that goes to $G$ instead of $S$.

These results allow us to verify the degree to which each class is sensitive to its respective parameters. For instance, the direction in which $m$ affects the $R$ class remains the same as we allow time to progress, but the magnitude of its effect varies wildly. In the first year it barely registers on the scale, but suddenly increases for the next few years.

## 4.5 Global Sensitivity Analysis

### 4.5.1 Money Equation Analysis

We chose a normal distribution for $d, r, \delta$ and $p_{max}$ because these values were derived from data and found to be the true means. The parameters $L$, $n$, and $f$ are uniformly distributed due to lack of data to better estimate the value of these parameters. $F_0$ is uniformly distributed from 0 to 1500, the total population size of a medium sized local prison center. The parameter $q_t$ has a uniform distribution since it has set values within a set range. $k$ is set to the formula for $k_{max}$, as the histogram where $k$ is set to $k_{min}$ are relatively the same.

Figures 9 and 10 only show the sensitivity of the third quarter for each case scenario. Only the third quarter is shown because the histograms for the second, third and fourth quarters were approximately the same, the exception being the first quarter where $a$ was not statistically significant.
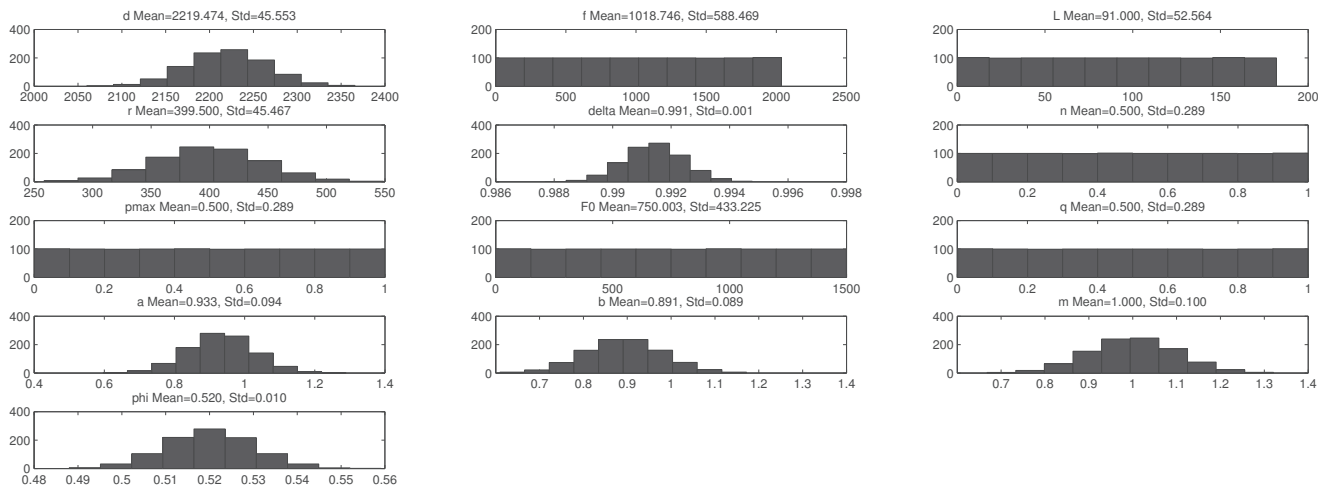


Figure 9: Parameter distributions used to find global sensitivity of the money equation at the third quarter
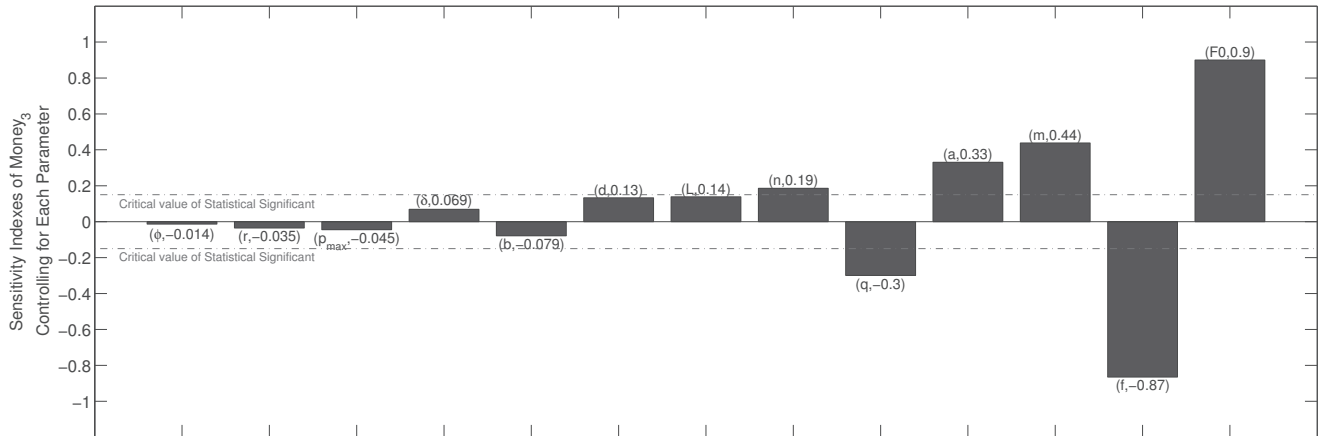
Figure 10: Sensitivity indexes of the Money equation at the third quarter

The global sensitivity analysis confirms that the Money Equation, which is an extension of the Profit Equation, is sensitive to $d$, $f$, $m$, and $a$. Globally, the Money Equation is also sensitive to $L$, the amount of money the prison is making from the labor of prisoners, $q_t$, the proportion of time inmates spend in the reform program, and $r$, the cost of the reform program. Money is also sensitive to $F_0$, the initial number of first time offenders in the prison.

### 4.5.2 Incarceration Difference Equation Analysis

Since the sensitivity plots after the second quarter all have the same shape and order of parameter importance, we show below the global sensitivity indexes for only one $F_t$, $R_t$, and $S_t$. For example, Figure 16 shows the global sensitivity of the $S_t$ equation at time three with respect to its parameters. The distributions of the parameters are shown in Figure 15. The distribution of $a, b, \phi$, and $m$ are normal because our data is able to give us an estimation of the true mean for these parameters. The half-saturation level for $p(q_t)$, $n$, has a uniform distribution because we do not have an accurate way to guess what the shape of this distribution looks like. The effectiveness of the reform program, $p_{max}$, and the allocation of prisoner's time, $q_t$, have a uniform distribution because their values are bounded between 0 and 1. The initial first-time offender population, $F_0$, also has a uniform distribution because we do not know what this value is prison to prison.
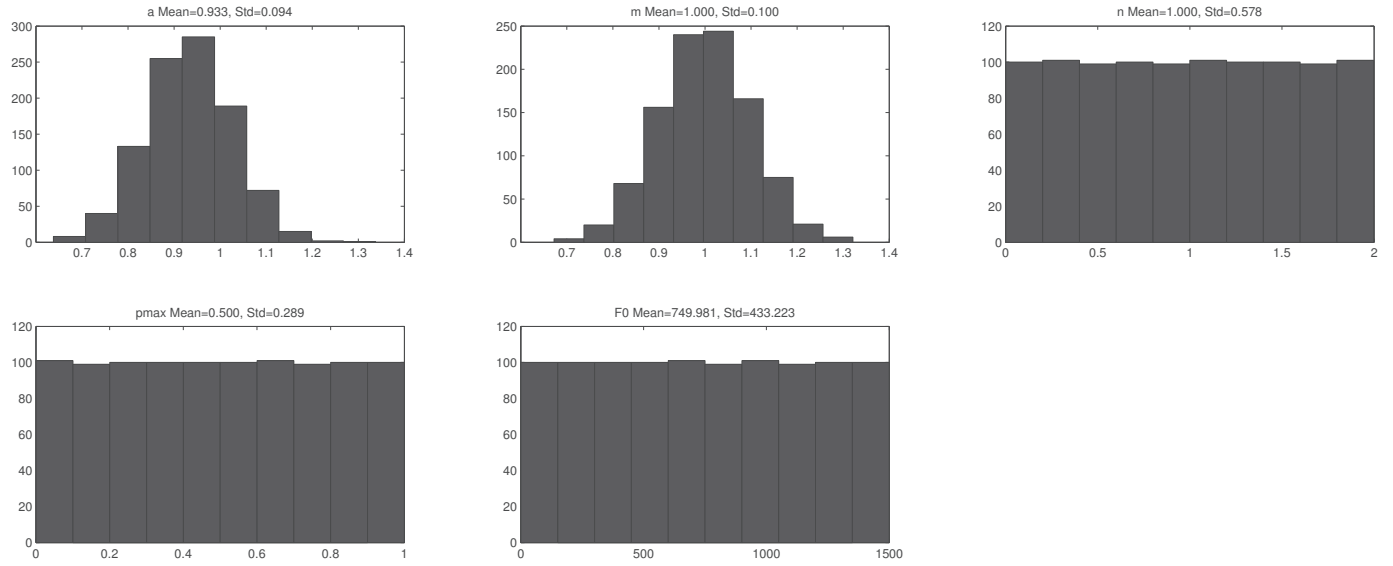
Figure 11: Parameter distributions used to find global sensitivity of $F_3$
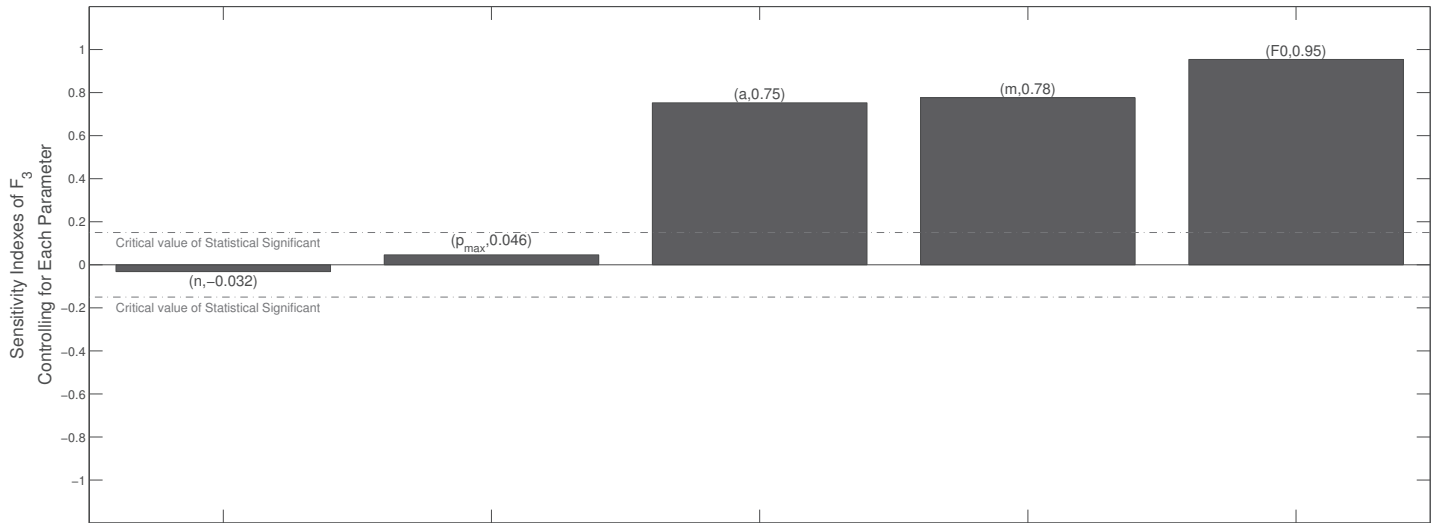


Figure 12: Sensitivity indexes of $F_3$
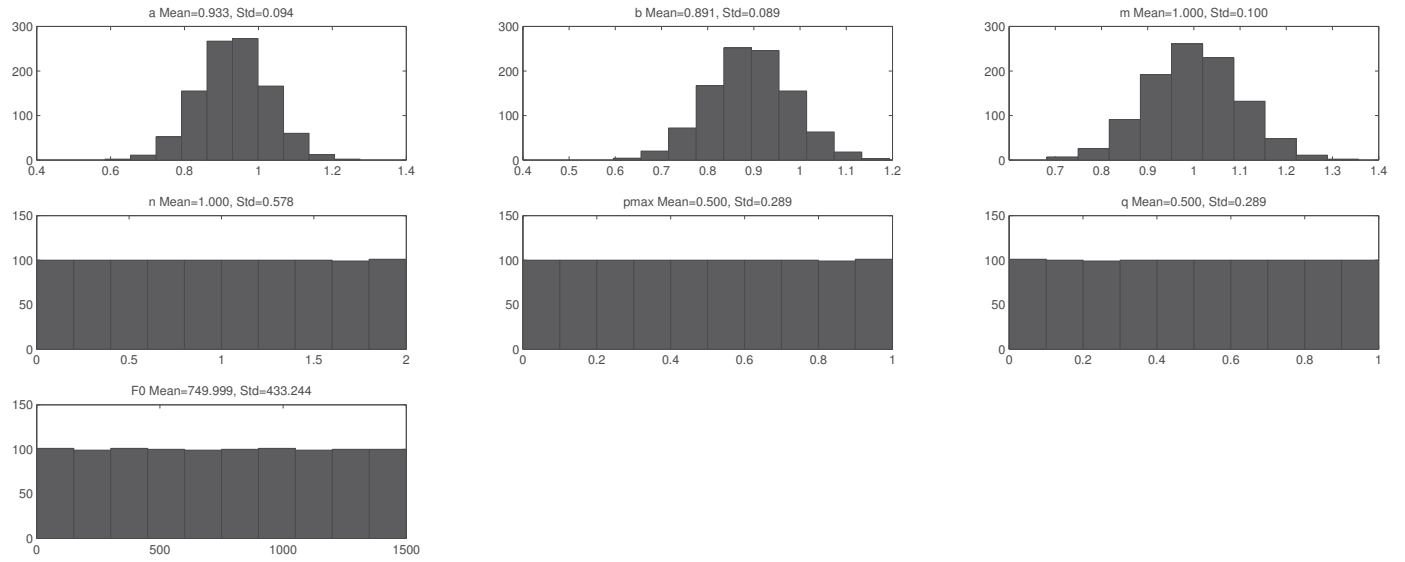
Figure 13: Parameter distributions used to find global sensitivity of $R_3$
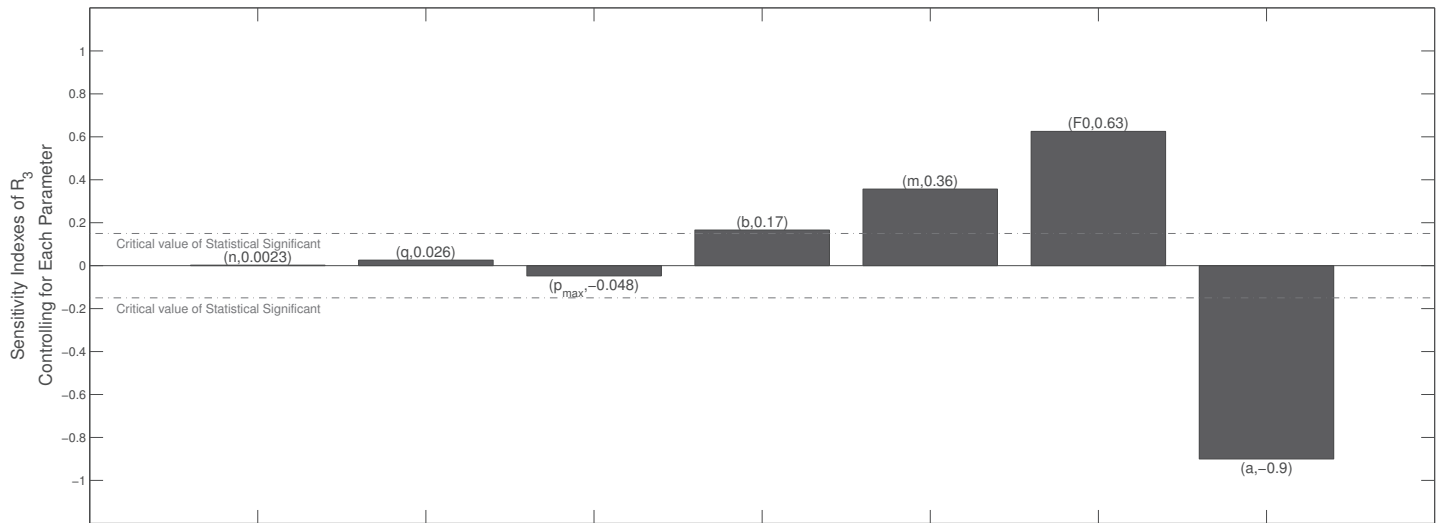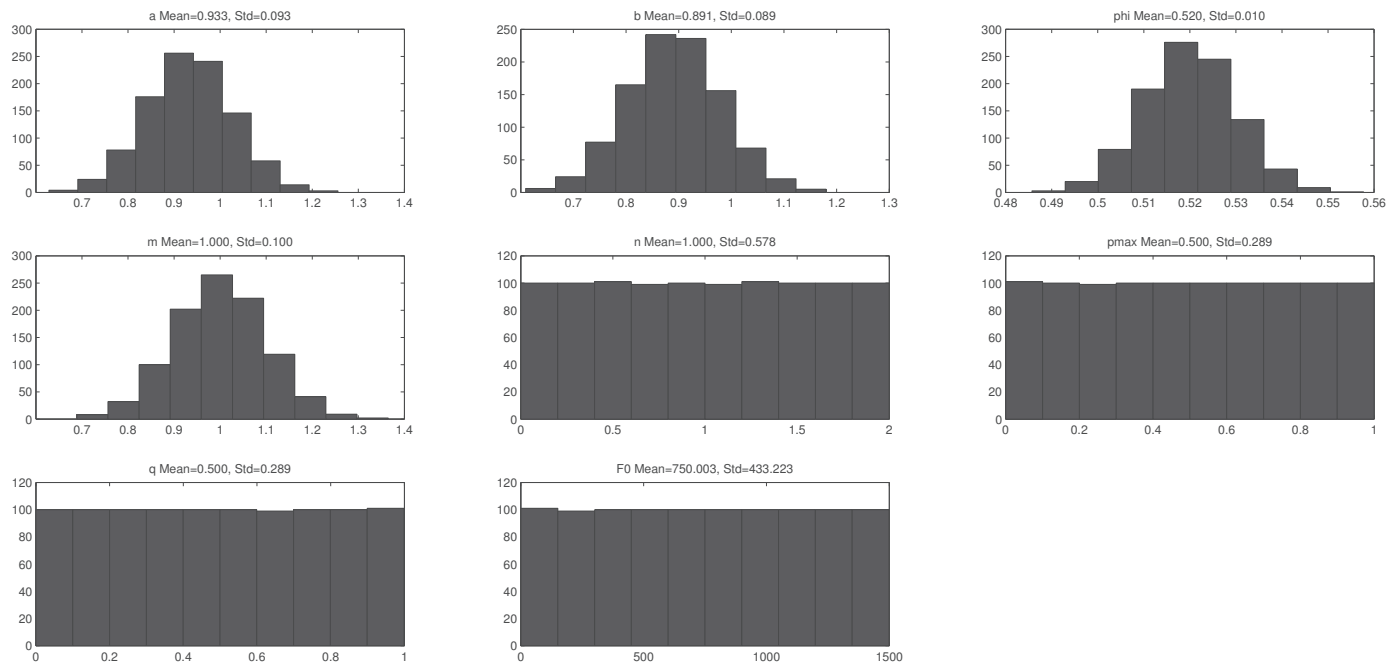


Figure 14: Sensitivity indexes of $R_3$

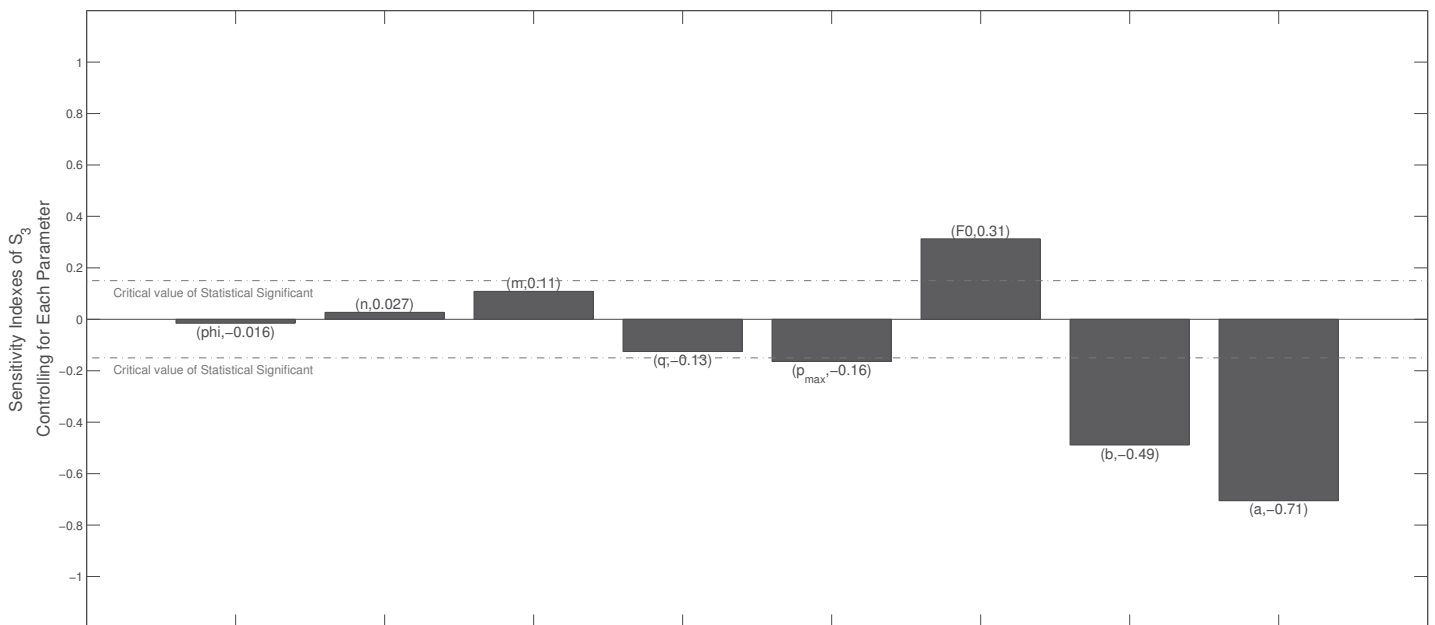Figure 15: Parameter distributions used to find global sensitivity of $S_3$



Figure 16: Sensitivity indexes of $S_3$

The global sensitivity of $F$, $R$, and $S$ confirmed the trend found in the local sensitivity analysis. For example, a one percent increase in $a$ or $b$ decreases the value of $S$. All classes are sensitive to the initial first-time offender population, $F_0$. Therefore it would be very

important to have accurate information on $F_0$ before using this model to make predictions about a prison. In addition to this, the global sensitivity showed that $S$ is sensitive not only to the $q_t$ of that time step, but also to the $p_{max}$. Therefore, the number of second time offenders is dependent on the success of the reform program.

# 5    Discussion

Utilizing the adaptive system, we were able to analyze how prisons may maximize profit as they adjust the allocation of prisoners' time between reform and labor programs. Although our model takes this problem from the perspective of a single prison, we can draw pertinent conclusions for the state as well. As the effectiveness of the program increases from $p_{max} = 0.1$ to 0.3, the incentive money provided by the state per day drops significantly, from \$12.62 to \$1.53 respectively. This effect was confirmed by the global sensitivity analysis of the Money equation, which showed an inverse relationship to $p_{max}$. This means more effective reform programs are in the interest of the state since they provide a more cost effective strategy in reducing recidivism. On the other hand, the prison is highly motivated to reduce the effectiveness of this reform program. This can be seen in the equation for the state incentive, $k$, where every slight decrease in the effectiveness of the program, $p_{max}$, has a large impact on increasing the incentive. When $p_{max} = 0.1$ the total money earned by the prison is more than double that earned when $p_{max} = 0.3$, which translates to around \$20 million more. This is due to the fact that when a reform program is less effective, it reduces recidivism less, which in turn makes it less able to command large incentives from the state. The state has unintentionally established a perverse incentive for the prison to reduce the effectiveness of reform programs. In order to motivate prisons to raise reform quality, the state may need to establish either strict standards or provide an additional monetary incentive for employing a more effective program.

One of the more plausible ways prisons can increase gross income is by reducing the total cost of each prisoner, $f$. This is due to the difference between the currently offered state per diem rate, $d$ and $f$, being pure profit. It is here where most sheriffs of local prisons are making the majority of their profit. Small cuts on a few expenditures and the housing of a few more inmates will only add to the total earnings. In the equations for $k_{min}$ and $k_{max}$ in Section 4.2, as the value of $f$ decreases the resulting boundaries increase for each $q_t$. This means that as prisons lower costs a larger incentive is required to obtain the same reduction in recidivism. In addition, the money earned from prisoners' labor has a large effect on how state incentives change $\bar{q}_t$. When all other parameters are held constant, a prison with no labor system in place, $L = 0$, will allocate more of the average prisoner's time into the reform program. As the profit of an individual prison increases, the amount of incentive that motivates them to employ the reform program increases as well. Therefore, when determining optimal incentive values it is important for the state to consider what the fixed cost per prisoner is on a case by case basis.

As seen in Figure 3, the incentives provided by the state have diminishing returns in how they reduce relative recidivism. It is important to note that this may be a reflection of the structure of $p(q_t)$, which is designed such that the effectiveness of the reform program has diminishing returns. Another artifact of the model is the manner in which $q_t^*$ fluctuates

every four quarters in Figures 5 and 6. This is due to the initial money equation using $\delta$ to account for the depreciating value of future profits. Consequently, this forces the prison to increase the allocation of time that prisoners spend in reform programs to maintain the same profit in the following three quarters. The magnitude of the fluctuation in $q_t^*$ increases as the cost of the reform program decreases, from $\pm 0.003$ for the current $r$ value to $\pm 0.02$ when $r = 0$.

For future work, we could improve how $\Lambda$ is defined, adding fluctuations that reflect changes in crime prevalence or law enforcement strategies. The perspective of the state may be of interest, as the state has a monetary motivation to decrease recidivism. Reducing recidivism increases the number of taxpayers and individuals contributing to the economy, allowing reform programs to potentially pay for themselves in the long run. Optimizing such a system where incentive values yield monetary returns in the upcoming years as a function of increased recidivism could help find the optimal incentive value that serves the interest of the state. The interplay between the interests of the state versus those of private prisons could also be explored as an example of tit-for-tat game theory. Decisions to lower recidivism could be beneficial only to the state, only to the prison, or to both.

Another possible improvement is the addition of another class, third time offenders, representing a three strikes law system. This would allow second time offenders to participate in the reform program, changing the dynamics of recidivism in the system. This would also allow the time the simulation is run to be extended from 5 years to 10 or more years.

# 6    Acknowledgments

# References

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# 7 Appendix

## 7.1 Solving the Difference Equations

In order to solve

$$F_{t+1} = \left[ e^{-\gamma T} F_t + \Lambda \right] e^{-\mu T},$$

$$R_{t+1} = \left[ \left(1 - e^{-\gamma T}\right) F_t + e^{-\alpha T} R_t \right] e^{-\mu T},$$

$$S_{t+1} = \left[ \left(1 - p(q)\right)\left(1 - \phi\right)\left(1 - e^{-\alpha T}\right) R_t + S_t \right] e^{-\mu T},$$

$$G_{t+1} = \left[ \left(\phi + (1 - \phi)p(q)\right)\left(1 - e^{-\alpha T}\right) R_t + G_t \right] e^{-\mu T},$$

we recall

$$
\begin{aligned}
m &= e^{-\mu T}, \\
a &= e^{-\gamma T}, \\
b &= e^{-\alpha T}, \\
c(q) &= \left(1 - p(q)\right)\left(1 - \phi\right), \\
h(q) &= \left(\phi + (1 - \phi)p(q)\right),
\end{aligned}
$$

allowing us to arrive at

$$F_{t+1} = [aF_t + \Lambda]m,$$
$$R_{t+1} = [(1-a)F_t + bR_t]m,$$
$$S_{t+1} = [c(q)(1-b)R_t + S_t]m,$$
$$G_{t+1} = [h(q)(1-b)R_t + G_t]m.$$

We first solve the $F_t$ equation

$$F_1 = amF_0 + \Lambda m$$
$$F_2 = amF_1 + \Lambda m$$
$$\quad = am[amF_0 + \Lambda m] + \Lambda m$$
$$\quad = a^2m^2F_0 + \Lambda am^2 + \Lambda m$$
$$F_3 = amF_2 + \Lambda m$$
$$\quad = am[a^2m^2F_0 + \Lambda am^2 + \Lambda m] + \Lambda m$$
$$\quad = a^3m^3F_0 + \Lambda a^2m^3 + \Lambda am^2 + \Lambda m$$
$$F_t = (am)^tF_0 + \Lambda \sum_{i=0}^{t-1} a^i m^{i+1}$$
$$F_t = (am)^tF_0 + \Lambda m \sum_{i=0}^{t-1} (am)^i$$
$$F_t = (am)^tF_0 + \Lambda m \frac{1-(am)^t}{1-am}.$$

Next we can look at the $R_t$ equation

$$R_1 = (1-a)mF_0 + bmR_0,$$
$$R_2 = (1-a)mF_1 + bmR_1,$$
$$\quad = (1-a)mF_1 + bm[(1-a)mF_0 + bmR_0],$$
$$\quad = (1-a)mF_1 + (1-a)bm^2F_0 + b^2m^2R_0,$$
$$R_3 = (1-a)mF_2 + bmR_2,$$
$$\quad = (1-a)mF_2 + bm[(1-a)mF_1 + (1-a)bm^2F_0 + b^2m^2R_0],$$
$$\quad = (1-a)mF_2 + (1-a)bm^2F_1 + (1-a)b^2m^3F_0 + b^3m^3R_0,$$
$$R_t = (bm)^tR_0 + (1-a)\sum_{i=0}^{t-1} b^{t-1-i}m^{t-i}F_i,$$
$$R_t = (bm)^tR_0 + (1-a)\sum_{i=0}^{t-1} b^{t-1-i}m^{t-i}\left((am)^iF_0 + \Lambda m\frac{1-(am)^i}{1-am}\right),$$

and finally the $S_t$ equation

$$S_1 = c_0(1-b)mR_0 + mS_0,$$
$$S_2 = c_1(1-b)mR_1 + mS_1,$$
$$= c_1(1-b)mR_1 + m(c_0(1-b)mR_0 + mS_0),$$
$$= c_1(1-b)mR_1 + c_0m^2(1-b)R_0 + m^2S_0,$$
$$S_3 = c_2(1-b)mR_2 + mS_2,$$
$$= c_2(1-b)mR_2 + m(c_1(1-b)mR_1 + c_0m^2(1-b)R_0 + m^2S_0),$$
$$= c_2(1-b)mR_2 + c_1m^2(1-b)R_1 + c_0m^3(1-b)R_0 + m^3S_0,$$
$$S_t = m^tS_0 + (1-b)\sum_{i=0}^{t-1}c(q_i)m^{t-i}R_i,$$
$$S_t = m^tS_0 + (1-b)\sum_{i=0}^{t-1}c(q_i)m^{t-i}\left((bm)^iR_0 + (1-a)\sum_{k=0}^{i-1}b^{i-1-k}m^{i-k}\left((am)^kF_0 + \Lambda m\frac{1-(am)^k}{1-am}\right)\right).$$

We can now take the derivative of $S_t$ and get

$$\frac{\partial S_t}{\partial q_t} = (1-b)\sum_{i=0}^{t-1}\frac{\partial c(q_i)}{\partial q_t}m^{t-i}\left((bm)^iR_0 + (1-a)\sum_{k=0}^{i-1}b^{i-1-k}m^{i-k}\left((am)^kF_0 + \Lambda m\frac{1-(am)^k}{1-am}\right)\right),$$
$$= 0.$$

Finally we solve for $G_t$

$$G_1 = h_0(1-b)mR_0 + mG_0,$$
$$G_2 = h_1(1-b)mR_1 + mG_1,$$
$$= h_1(1-b)mR_1 + m(h_0(1-b)mR_0 + mG_0),$$
$$= h_1(1-b)mR_1 + h_0m^2(1-b)R_0 + m^2G_0,$$
$$G_3 = h_2(1-b)mR_2 + mG_2,$$
$$= h_2(1-b)mR_2 + m(h_1(1-b)mR_1 + h_0m^2(1-b)R_0 + m^2G_0),$$
$$= h_2(1-b)mR_2 + h_1m^2(1-b)R_1 + h_0m^3(1-b)R_0 + m^3G_0,$$
$$G_t = m^tG_0 + (1-b)\sum_{i=0}^{t-1}h(q_i)m^{t-i}R_i,$$
$$G_t = m^tG_0 + (1-b)\sum_{i=0}^{t-1}h(q_i)m^{t-i}\left((bm)^iR_0 + (1-a)\sum_{k=0}^{i-1}b^{i-1-k}m^{i-k}\left((am)^kF_0 + \Lambda m\frac{1-(am)^k}{1-am}\right)\right).$$

## 7.2 Equilibria of the Incarceration Dynamics

The following equilibria only exist when $q_t = q$ is constant.
We can now find $F_\infty$,

$$F_\infty = amF_\infty + \Lambda m,$$

$$F_\infty = \frac{\Lambda m}{(1 - am)},$$

And $R_\infty$

$$R_\infty = (1 - a)mF_\infty + bmR_\infty,$$

$$R_\infty = \frac{(1 - a)mF_\infty}{(1 - bm)},$$

Then $S_\infty$,

$$S_\infty = c(q)(1 - b)mR_\infty + mS_\infty,$$

$$S_\infty = \frac{c(q)(1 - b)mR_\infty}{(1 - m)},$$

And finally $G_\infty$

$$G_t = h(q)(1 - b)mR_\infty + mG_\infty,$$

$$G_\infty = \frac{h(q)(1 - b)mR_\infty}{(1 - m)}.$$

## 7.3 Boundary Conditions for k

Using the parameter estimates we can compute the boundary conditions for $k$ when we are at equilibrium for both $F_t$ and $R_t$. For this particular calculation we will be using $L = 182$, $\Lambda = 50$, $r = 392.21$, $f = 2037.50$, $d = 2219.50$, and for a $p_{max} = .2$. We denote $q_\infty$ as the value of $q_t$ when it is calculated at equilibrium.

For $q_\infty = 0$,

$$
\begin{aligned}
k_{min} &= \frac{n(L_q + r_q)}{(n + 1)p_{max}} + \frac{(d_q - f_q + L_q)(1 - b)(1 - \phi)R_\infty \sum_{i=1}^{y}(\delta m)^i}{F_\infty} \\
&= \frac{1(182 + 392.21)}{(1 + 1).2} + \frac{(2219.5 - 2037.5 + 182)(1 - 0.8914)(1 - 0.52)R_\infty \sum_{i=1}^{y}((0.9914)(0.9997))^i}{F_\infty}
\end{aligned}
$$

For $q_\infty = 1$,

$$k_{max} = \frac{(n+1)(L_q + r_q)}{np_{max}} + \frac{(d_q - f_q + L_q)(1-b)(1-\phi)R_\infty \sum_{i=1}^{y}(\delta m)^i}{F_\infty}$$

$$= \frac{(2)(182 + 392.21)}{1(0.2)} + \frac{(2219.5 - 2037.5 + 182)(1 - 0.8914)(1 - 0.52)R_\infty \sum_{i=1}^{y}((0.9914)(0.9997))^i}{F_\infty}$$

Since we know that

$$F_\infty = \frac{\Lambda m}{(1 - am)}$$

$$= \frac{50(0.9997)}{1 - (0.933)(0.9997)}$$

$$= 743.5757$$

$$R_\infty = \frac{(1 - a)mF_\infty}{(1 - bm)}$$

$$= \frac{(1 - 0.933)(0.9997)(743.5757)}{(1 - (0.8914)(0.9997))}$$

$$= 457.16634,$$

it follows that

For $q_\infty = 0$,

$$k_{min} = 1435.57728 + \frac{8677.24907(2.94705)}{743.5757}$$

$$= 1435.57728 + 34.391$$

$$\approx 1469.97$$

For $q_\infty = 1$,

$$k_{max} = 5742.10 + \frac{8677.24907(2.94705)}{743.5757}$$

$$= 5742.10 + 34.391$$

$$\approx 5776.49$$

## 7.4 Condition on $q_t$

The general form of $q_t^*$ is:

$$q_t^* = -n \pm \sqrt{n(n+1)p_{max}\left(\frac{kF_t - (d - f + L)(1-b)(1-\phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t(L + r)}\right)}.$$

In order for $q_t^*$ to be non negative we only consider the $(+)$ root and require that:

$$n \leq \sqrt{n(n+1)p_{max}\left(\frac{kF_t - (d - f + L)(1 - b)(1 - \phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t(L + r)}\right)}$$

so

$$n^2 \leq n(n+1)p_{max}\left(\frac{kF_t - (d - f + L)(1 - b)(1 - \phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t(L + r)}\right)$$

then

$$\frac{n}{(n+1)p_{max}} \leq \left(\frac{kF_t - (d - f + L)(1 - b)(1 - \phi)R_t \sum_{i=1}^{y}(\delta m)^i}{F_t(L + r)}\right)$$

since

$$\frac{(n+1)p_{max}}{n} = \left.\frac{\partial p(q_t)}{\partial q_t}\right|_{q_t=0}$$

$$\left.\frac{\partial p(q_t)}{\partial q_t}\right|_{q_t=0} \geq \frac{F_t(L + r)}{kF_t - (d - f + L)(1 - b)(1 - \phi)R_t \sum_{i=1}^{y}(\delta m)^i}$$

## 7.5   Profit Sensitivity Analysis

In this section we will find all the sensitivity indices for all the parameters and state variables in the profit equation, redefined below.

$$\text{Profit}_t = \left[(d - f + L)S_t + \left(L(1 - q_t) + kp(q_t) + d - f - rq_t\right)F_t\right]T$$

The sensitivity indices for all the parameters of the Profit equation are given by

$$\text{Sensitivity}(\text{Profit}, d) = (S_t + F_t)T\left(\frac{d}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, f) = (-S_t - F_t)T\left(\frac{f}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, L) = (S_t + (1 - q_t)F_t)T\left(\frac{L}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, r) = (-q_t F_t)T\left(\frac{r}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, k) = (p(q_t)F_t)T\left(\frac{k}{\text{Profit}_t}\right),$$

$$= \left(\left(p_{max}(n + 1)\frac{q_t}{q_t + n}\right)F_t\right)T\left(\frac{k}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, a) = \left[(d - f + L)\left(\frac{\partial S_t}{\partial a}\right) + ((d - f + L(1 - q) - rq + k)\left(\frac{\partial F_t}{\partial a}\right)\right]T\left(\frac{a}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, b) = (d - f + L)\left(\frac{\partial S_t}{\partial b}\right)T\left(\frac{b}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, m) = \left[ (d - f + L)\left(\frac{\partial S_t}{\partial m}\right) + ((d - f + L(1 - q) - rq + k)\left(\frac{\partial F_t}{\partial m}\right) \right] T\left(\frac{m}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, \Lambda) = \left[ (d - f + L)\left(\frac{\partial S_t}{\partial \Lambda}\right) + ((d - f + L(1 - q) - rq + k)\left(\frac{\partial F_t}{\partial \Lambda}\right) \right] T\left(\frac{\Lambda}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, \phi) = (d - f + L)\left(\frac{\partial S_t}{\partial \phi}\right) T\left(\frac{\phi}{\text{Profit}_t}\right).$$

The sensitivity indices with respect to the state variables of the Profit equation are given by:

$$\text{Sensitivity}(\text{Profit}, S_t) = (d - f + L)T\left(\frac{S_t}{\text{Profit}_t}\right),$$

$$\text{Sensitivity}(\text{Profit}, F_t) = ((d - f + L(1 - q) - rq + k)T\left(\frac{F_t}{\text{Profit}_t}\right).$$

## 7.6   Migration Class $(F_t, R_t, S_t, G_t)$ Sensitivity Analysis

The sensitivity indices for the state variables are:

$$\text{Sensitivity}(F_t, m) = \left( ta^t m^{t-1} F_0 + \Lambda \sum_{i=0}^{t-1}(am)^i + \Lambda m \sum_{i=0}^{t-1} ia^i m^{i-1} \right)\left(\frac{m}{F_t}\right),$$

$$\text{Sensitivity}(F_t, a) = \left( ta^{t-1} m^t F_0 + \Lambda m \sum_{i=0}^{t-1} ia^{i-1} m^i \right)\left(\frac{a}{F_t}\right),$$

$$\text{Sensitivity}(F_t, \Lambda) = m\frac{1 - (am)^t}{1 - am}\left(\frac{\Lambda}{F_t}\right),$$

$$\text{Sensitivity}(R_t, m) = \left( tb^t m^{t-1} R_0 + (1 - a)\left( \sum_{i=0}^{t-1}(t - i)(bm)^{t-1-i} F_i + \sum_{i=0}^{t-1} b^{t-1-i} m^{t-i}\left(\frac{\partial F_i}{\partial m}\right) \right) \right)\left(\frac{m}{R_t}\right),$$

$$\text{Sensitivity}(R_t, a) = \left( -\sum_{i=0}^{t-1} b^{t-i-1} m^{t-i} F_i + (1 - a)\sum_{i=0}^{t-1} b^{t-1-i} m^{t-i}\left(\frac{\partial F_i}{\partial a}\right) \right)\left(\frac{a}{R_t}\right),$$

$$\text{Sensitivity}(R_t, b) = \left( tb^{t-1} m^t R_0 + (1 - a)\sum_{i=0}^{t-1}(t - i - 1)m^{t-i} b^{t-2-i} F_i \right)\left(\frac{b}{R_t}\right),$$

$$\text{Sensitivity}(R_t, \Lambda) = \left( (1 - a)\sum_{i=0}^{t-1} m^{t-i} b^{t-1-i}\left(\frac{\partial F_i}{\partial \Lambda}\right) \right)\left(\frac{\Lambda}{R_t}\right),$$

$$\text{Sensitivity}(S_t, m) = \left( tm^{t-1} S_0 + (1 - b)\left( \sum_{i=0}^{t-1}(t - i)c(q_i)m^{t-1-i} R_i + \sum_{i=0}^{t-1} c(q_i)m^{t-i}\left(\frac{\partial R_i}{\partial m}\right) \right) \right)\left(\frac{m}{S_t}\right),$$

$$\text{Sensitivity}(S_t, a) = \left( (1 - b) \sum_{i=0}^{t-1} c(q_i) m^{t-i} \left( \frac{\partial R_i}{\partial a} \right) \right) \left( \frac{a}{S_t} \right) ,$$

$$\text{Sensitivity}(S_t, b) = \left( - \sum_{i=0}^{t-1} c(q_i) m^{t-i} R_i + (1 - b) \sum_{i=0}^{t-1} c(q_i) m^{t-i} \left( \frac{\partial R_i}{\partial b} \right) \right) \left( \frac{b}{S_t} \right) ,$$

$$\text{Sensitivity}(S_t, \phi) = \left( (1 - b) \sum_{i=0}^{t-1} \left( p(q_i) - 1 \right) m^{t-i} R_i \right) \left( \frac{\phi}{S_t} \right) ,$$

$$\text{Sensitivity}(S_t, \Lambda) = \left( (1 - b) \sum_{i=0}^{t-1} c(q_i) m^{t-i} \left( \frac{\partial R_i}{\partial \Lambda} \right) \right) \left( \frac{\Lambda}{S_t} \right) ,$$

$$\text{Sensitivity}(S_t, p_{max}) = \left( (1 - b)(1 - \phi) \sum_{i=0}^{t-1} m^{t-i} R_i \left( 1 - \frac{q_i(n+1)}{q_i + n} \right) \right) \left( \frac{p_{max}}{S_t} \right) ,$$

$$\text{Sensitivity}(S_t, n) = \left( (1 - b)(1 - \phi) \sum_{i=0}^{t-1} m^{t-i} R_i \left( 1 - p_{max} q_i \frac{q_i - 1}{(q_i + n)^2} \right) \right) \left( \frac{n}{S_t} \right) .$$
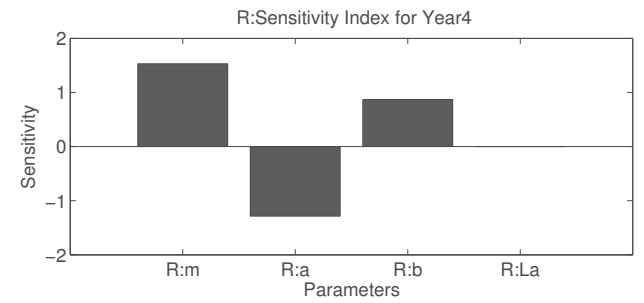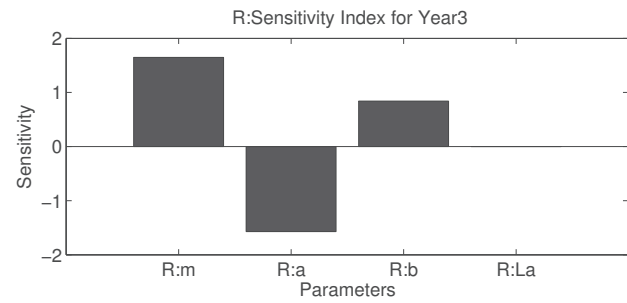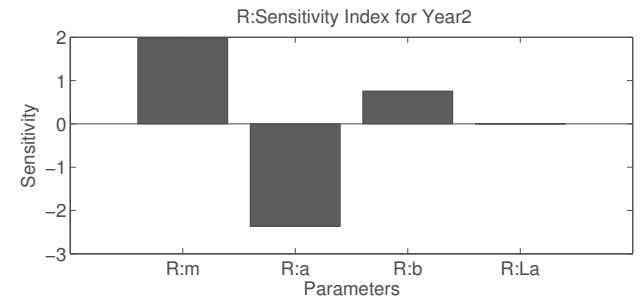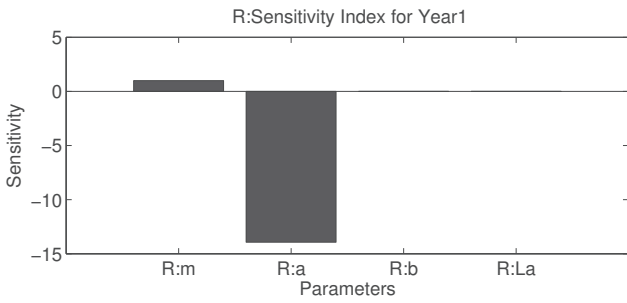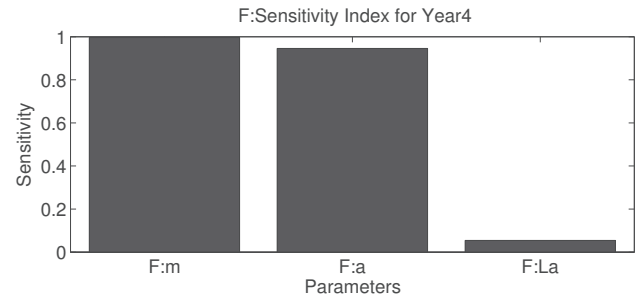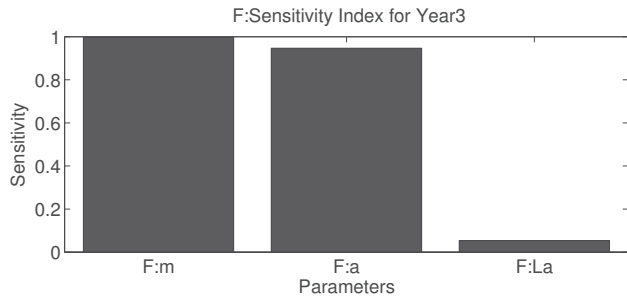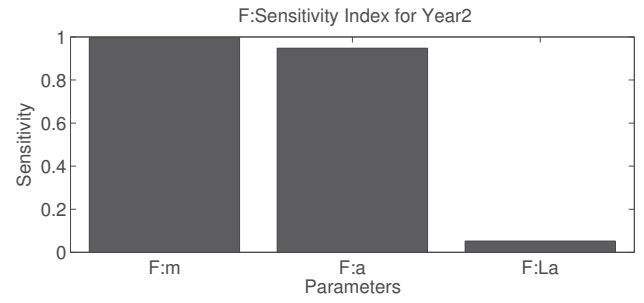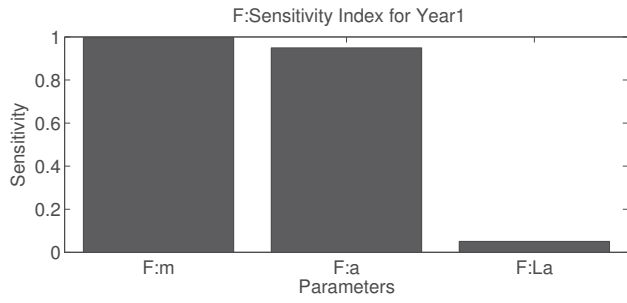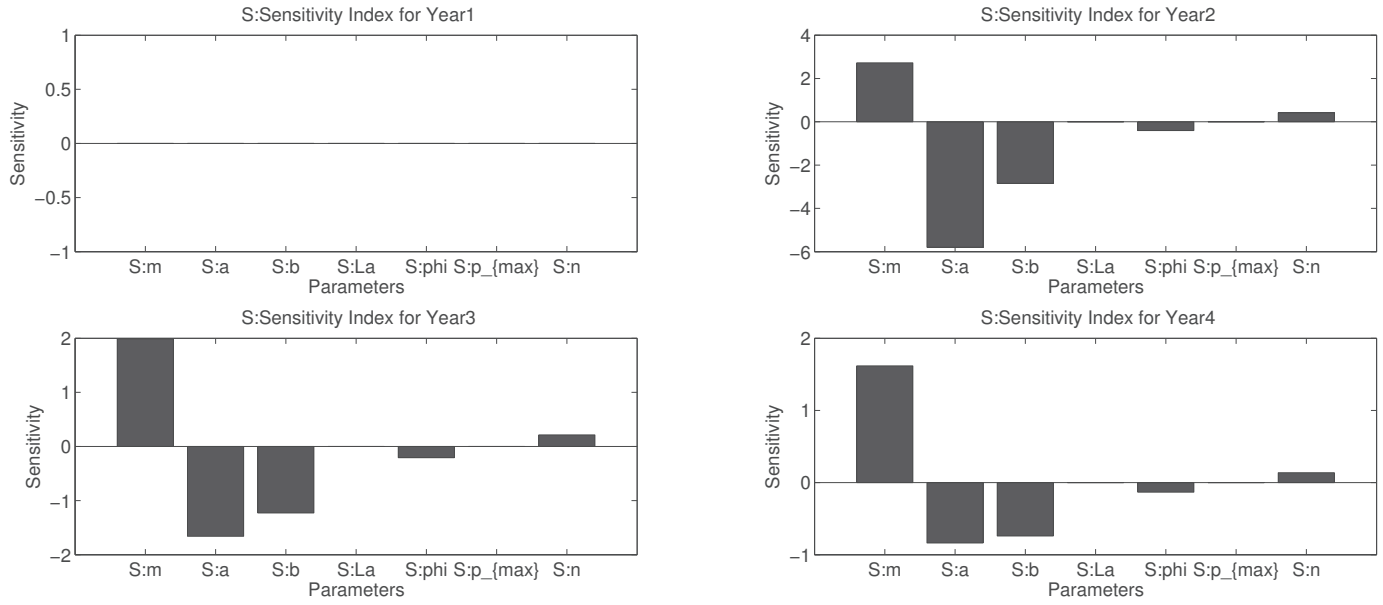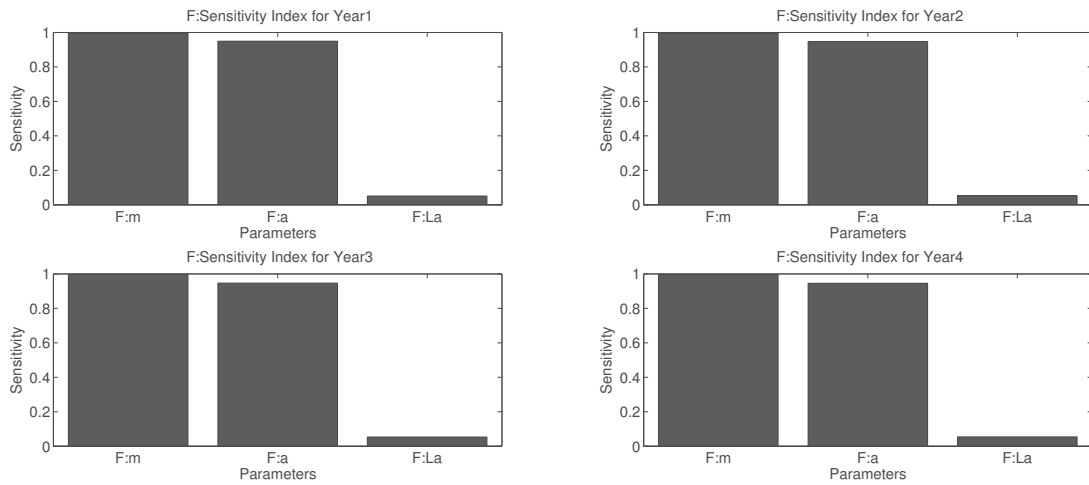
## 7.7  Sensitivity

Figure 17: Sensitivity indices of $F$, $R$, and $S$ respectively when $p_{max} = 0.1$
Note scales are different to show the difference in relative magnitude in each separate year.
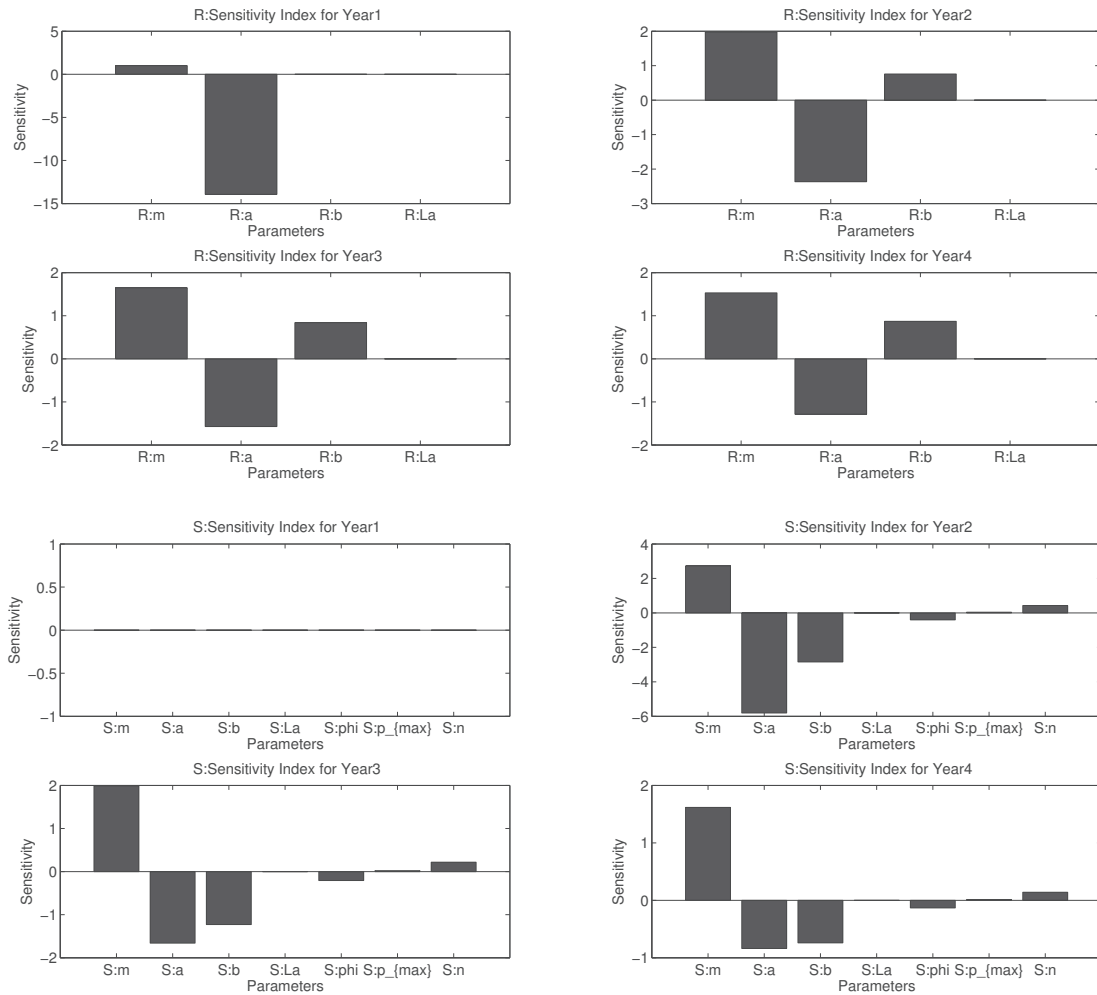
Figure 18: Sensitivity indices of $F$, $R$, and $S$ respectively when $p_{max} = 0.2$
Note scales are different to show the difference in relative magnitude in each separate year.