

# Evaluating Cost Effective Control Strategies for dealing with Beaver Infestation in Tierra del Fuego.

Diego Vicencio Morales<sup>1</sup>, Juan Meléndez<sup>2</sup>, and Anuj Mubayi<sup>2</sup>

<sup>1</sup>Universidad Técnica Federico Santa María

<sup>2</sup>Arizona State University

## Abstract

In 1947, the non-native species of North American beaver was introduced in the southern tip of South America, with the goal to encourage fur industry. Since then, beaver population has spread and grown many fold, resulting in a huge challenge to control its population in Tierra del Fuego, covering regions in both Chile and Argentina. Primary reasons for their rapid spread in part include good ecological conditions and lack of natural predators, thus causing huge ecological damage in the region. This study uses data from Tierra del Fuego on growth rates, dispersal rate of beaver population and types of available controls to develop dynamic models of population spread. These models are used to design effective population control policies that include trapping and hunting. Monetary cost for the implementation of these controls are considered in order to optimize costs and reduction policies for development of long lasting strategies.

*Index terms*— Infestation, Diffusion, Control, Invasive Species, Tierra del Fuego

## 1 Introduction

The economic globalization has allowed the development of economic infrastructure all around the world, which in turn has developed multiple new markets and industries. These development-related economic impulse also led to the creation of animal product based industries in non-natural environments. For example, the introduction of zoological species in zones outside of their natural habitat has often been attempted to encourage certain industrial products. This man-made intervention has caused huge ecological problems, such as the damage to an environment caused by a foreign species which are not native to the local ecosystem. Attempting to contain and protect the local flora and fauna, and even human population, from these

invaders has also created long-term challenges for governments and control agencies around the world. [10][12][13].

Infestation of invasive foreign species has affected countries such as Chile, a country which its great variety of climates allows many unique conditions for the development of a rich variety of flora and fauna. However, this variety of climates also has led, by ignorance or misled attempts to create new animal industries, to the introduction of invasive species, which has been the cause of many problems and devastation for the local environment [7]. Cases such as the introduction of wild boars in the south of Chile (introduced in Chile in the 40s by german immigrants) [16], rabbits all along the center and south of Chile in 1884 [7], and in Tierra del Fuego in 1936 [6], can be considered emblematic examples of this type of infestation phenomena of species introduced in a foreign environment. In the case of rabbits, for example, numbers got staggering. Four rabbits were introduced in 1936, and due to good conditions and lack of predators, the numbers were estimated in 30,000,000 by 1954 [6]. Multiple control methods were attempted, such as the introduction of foxes and encouraging hunting activities, and even, poison gas, all without success. In the end, the introduction of the myxomatosis virus was the method which helped to crash the rabbit population growth [6]. Although rabbit population seems to be controlled in Tierra del Fuego, the problem cannot be considered completely solved, as the example of Australia shows, in which, multiple control policies have been attempted, and rabbit population control remain a considerable problem for the local government. [20]

In the present study, the focus will be on the current north-american beaver infestation which affects Tierra del Fuego. This geographical zone, located in the southern extreme of Chile, is a very remote area, with difficult access and small population. Beaver infestation is currently considered a huge ecological problem in the area, and has attracted the attention of the Chilean and Argentinian governments [11], both of which are currently planning a joint effort to conduct the eradication of this infestation.[4]

The north american beaver(*Castor Canadensis*), was introduced in Tierra del Fuego, in the Argentinian sector, in 1947, around the area of lake Fagnano, in an unsuccessful attempt to create a fur industry in the area [14] [11] [19]. Initially, 25 beavers were released, and 70 years later, the beavers have grown into an infestation in the area. Beavers have expanded geographically not just in the main island of Tierra del Fuego, but also to other surrounding islands, such as Isla Navarino, Isla Dawson, and now reported to be present in the american

continent. The population is now estimated between 150,000 and 200,000, which, as some news sources have described, greater than the human population (134.000) in the Argentinian zone of Tierra del Fuego (See: [3])

The north american beaver has been described as an engineer of the rivers. Its natural behavior is to set colonies in river sectors, near a dam built with wooden materials, which serves as a refuge against predators, and source of food. In its natural environment, trees have co-evolved with the beaver, and fortunately most of the natural flora consumed by the beaver in its natural environment has the ability to regrow from stumps left by the beavers. The local flora of Tierra del Fuego does not have such characteristics, and this has led to huge ecological devastation in native flora [14] [19]. Beaver population has been considered an infestation in some areas of their natural habitat, North America, given the reduced number of some of their natural predators (bears and wolves), and studies have been done to design control policies for such cases [1], but to our knowledge, the problem never has reached the staggering numbers the infestation has in Tierra del Fuego.

The growth of beaver population during the last 70 years since its introduction is characterized works in [14] and [19] as a typical situation in which a population grows without a predator, in a favorable environment. The beavers also seem to have adjusted pretty well in the local area, consuming a large number of local flora in the area, in particular *lengas*, a quite common local tree in the Patagonia and Tierra del Fuego, which seems to be the factor leading its spreading [19].

Although large literature exists that attempt to address this problem (see for example: [2][11][14][19]), and studies warning about the dangers of further expansion (see for example, [17]), and governments announcements about the intention to deal with the problem, as of 2017, seems that just recently both the Argentinian and Chilean governments are announcing concrete plans for dealing with the problem [4].

The objective of our work was to study population dynamics and spread of Beavers, and to develop cost-effective trapping strategies using data from Tierra del Fuego, with the aim of reducing both the damages caused by the beaver infestation, and reducing the cost associated with trapping practices. To do so, we will use mathematical and tools such as the used in [1] and [5], but with the novelty of using different cost estimates generated with the data of Tierra del Fuego, obtained in works such as [18], which will allow us to use different types of cost functions for our analysis, and also, we will add spatial considerations for the application

of control policies, based in the data of Tierra del Fuego and different strategic considerations for beaver-trapping.

## 2 Methods and Model Description

### 2.1 Data discussion and methods

Wallem et. al.[19] described the dynamics of Beaver expansion, although, as the text indicates, some caveats have to be considered, given that the topic has not been thoroughly investigated, and data used in their work was based on opinions and estimations by experts. According to the authors, the dynamics of beaver expansion are mainly consistent with expansion of an species with lack of predators. Thus, the spreading is described as a model of simple diffusion[15]. (An example of this kind of diffusion can be seen in [8])

Beaver diffusion rates are also estimated in Wallem et. al.[19]. Beavers are a territorial species. Families of five or six member tend to live near a dam, and the offspring are thrown out of the colony at age two or three, which leads them to form their own dams. Intruder beavers are not accepted in foreign colonies. These natural factors lead to spreading by lack of spaces available for colonization. Somewhat paradoxically, it seems diffusion is faster in steppes or sub=forest areas, which, according to [19], may be explained by depletion of resources. In dense forest areas, spread is slower, because reoccupation of colonies is common. In this work, for sake of simplicity of the mathematical analysis, constant diffusion rates are assumed, leaving the problem open for non-constant rates.

The methodology of beaver control considered for the present work is trapping, as suggested in works such as [1] [5]. Beaver trapping is the general name we use to describe a multiple array of practices used for trapping, capture or eliminate beavers from a sector. It involves a series of elements ranging from using individual or multiple traps, to more strong methods such as the use of explosives.

Also, and this is considered an open question, considering the success of the use of myxomatosis virus as a infestation control policy for rabbits in Tierra del Fuego [6], it seems that such an idea may be considered for beavers. During the present study, we couldn't find any example of control policy associated with such an idea, but it remains an open question for further scientific investigation, and mathematical modeling.

Local data suggested that the cost of implementing a trapping intervention couldn't be well

captured by just using studies of trapping strategies in other parts of the world, for example, using data as suggested in [1]. Thus, using local studies such as [18], we considered different costs estimations based on public opinion, with higher order polynomial functions to represent the sensitivity of such a variable. We considered in this work that more data in cost estimation should be gathered to design more effective cost estimations.

Our aim in the present work is to use the data related to beaver behavior, reproduction and population and costs associated with control policies and ecological costs, we intend to develop and use mathematical tools of Optimal Control theory to develop necessary conditions for the design of trapping strategies, and then, comparing these strategies using numerical methods to compare their effectiveness, in terms of reducing both the cost associated with policies and the ecological cost associated with beaver presence in Tierra del Fuego.

## 2.2 General Model of Beaver expansion and Control Problem

Thus, our problem can be expressed in general as the following: Given a beaver population described by  $N(x, t)$ , where  $x$  is a spatial variable in some compact set  $\Omega \in \mathbb{R}^n$ , and  $t$  is a time variable, is subjected to the following dynamic:

$$\begin{aligned}
 N_t &= dN_{xx} + f(x, N) - s(u, x, N) \\
 \partial N(x, t) &= 0 \\
 N(x, 0) &= g(x)
 \end{aligned}
 \tag{1}$$

Where:

Element	Description
$N(x, t)$	Beaver population at spot $x$ , at time $t$ .
$\partial N(x_1, t)$	Boundary condition for spreading.
$d$	Diffusion rate of beaver population.
$f(x, N)$	Growth rate of beaver population.
$u(x, t)$	Control, considered rate of beaver trapping at spot $x$ , at time $t$ .
$s(u, x, N)$	Rate of implementation of a control at position $x$ .

Table 1: Parameter Description.

Given this dynamic equation, our optimization problem is to minimize a cost function,

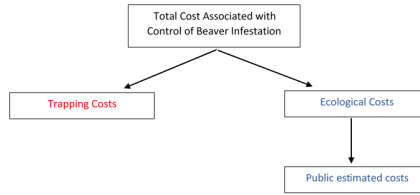


Figure 1: Costs considered. In red, trapping costs, in blue, ecological costs.

associated to a functional  $J[u]$  which depends on the control, and can be expressed as following:

$$\min_{u \in U} J[u] = \min_{u \in U} \int_0^T \int_{\Omega} F(u(x, t), N(x, t), x, t) dx dt$$

The function  $F$  is our cost function, which depends on the beaver population and the control. Using the model proposed in [1][5], we will consider in general direct cost functions, this is, we will consider functions associated with the monetary cost of beaver population damage, and also monetary cost of beaver control. For estimating the damage costs, in Tierra del Fuego, we will base our work mainly in the estimations done by [18], which costs are estimated using survey and public opinion data, rather than direct damage costs as is proposed in [1]. The diagram of costs considerations can be seen in Figure 1

### 3 Analysis

The aim of the present section is to develop the PDE problem of control in a general framework, to generate tools which will allow us to develop novel control strategies for the problem we setup in the previous section.

This chapter will be divided as follows. First, we will develop the result obtained in [5], as a means to review the main mathematical techniques used for the beaver control problem. Then, using this deduction and the techniques developed in it, we will extend that deduction in multiple ways, with the aim of providing of mathematical tools to develop novel control techniques for dealing the beaver infestation control problem.

The scheme of the results we will develop in this section is shown here:

Cost Function	Growth Function	Control
Quadratic[1][5]	Linear	All Space
General	Linear	All Space
General	Linear	Restricted at an interval.
General	Linear	Multiple controls
General	Logistic	All of the Above

Table 2: Scheme of Analysis Section

### 3.1 Case 1: Quadratic Cost Function. Linear Growth Function

For our initial optimization problem, we will assume the beaver population will spread in just one dimension. This assumption of a linear space shouldn't be considered so unrealistic, considering beaver populations usually make their refuges and dams near rivers. The Dirichlet initial conditions (the population at the border is 0) represent unsuitable conditions outside of the habitable area.

Consider an interval  $[x_1, x_2] \in \mathbb{R}$ , of length  $L$ , and a model of growth and spread of a beaver population by the following system:

$$\begin{aligned}
 N_t &= dN_{xx} + rN - uN \\
 N(x_1, t) &= N(x_2, t) = 0 \\
 N(x, 0) &= g(x)
 \end{aligned} \tag{2}$$

Note, here:

Element	Description
$N(x, t)$	Population density of beaver population at patch $x$ , at time $t$ .
$r$	Growth rate of population per time unit.
$f(x, N)$	Growth rate of beaver population.
$u(x, t)$	Control, considered rate of beaver trapping at spot $x$ , at time $t$ .
$u_{max}$	Upper bound of control policy
$g(x)$	Initial Condition.

Table 3: Parameters.

It is assumed our control takes values between  $[0, 1)$ , representing a culling ratio of population, and we have a value  $u_{max}$ , such that  $u(x, t) \in [0, u_{max}] \subset [0, 1)$ .

For simplicity, it is also assumed that our functions are on

$$L^2 [[0, T] \times [x_1, x_2], \mathbb{R}]$$

Our problem, thus, is defined as follows: Given the dynamic described in system (2), the problem is to minimize the following functional:

$$\min_{u \in U} \int_0^T \left[ \int_{x_1}^{x_2} \frac{1}{2} \gamma N^2(x, t) + cu^2(x, t)N(x, t) \right] dx dt \quad (3)$$

Where:

- **U**: Set of admissible controls (in principle, we will assume just to be piecewise continuous functions).
- **c**: Cost of control application.
- **$\gamma$** : Cost of environmental damage caused by density of beaver population.

The methodology described in Lenhart et. al. [5] will be used to find the optimal control for the problem defined by equations (2) and (3). The first step is to set what is called the sensitivity problem. Assume a trajectory  $N(x, t)$ , is associated with a certain control  $u(x, t)$ . In that case, a perturbed control  $u_\epsilon$  will be defined as:

$$u_\epsilon = u + l\epsilon,$$

where  $l$  is a perturbation function, and  $\epsilon$  is a positive real value (the sign of the perturbation is given, thus, by  $l$ ). Thus, the sensitivity function will be defined:

$$\psi(x, t) = \lim_{\epsilon \rightarrow 0} \frac{N_\epsilon(x, t) - N(x, t)}{\epsilon},$$

where  $N_\epsilon(x, t)$ , is the trajectory associated with the perturbed control. Our sensitivity function, thus, will have its own PDE associated with its dynamic, which can be deduced by summing the dynamics of  $N$  and  $N_\epsilon$ , dividing by  $\epsilon$ , and taking limit  $\epsilon \rightarrow 0$ . Notice that both  $N$  and  $N_\epsilon$  have the same function as initial condition. Thus, the dynamic of  $\psi$  is described by:



$$\begin{aligned}
\psi_t - d\psi_{xx} - r\psi + u\psi &= -lN \\
\psi(x_1, t) = \psi(x_2, t) &= 0 \\
\psi(x, 0) &= 0
\end{aligned} \tag{4}$$

Next, we define a linear operator  $L$ , such that  $L\psi = -lN$ . Our next step will be to find the adjoint operator  $L^*$  associated with  $L$ . As we said earlier, we're considering our functions in  $L^2$ , which is a Hilbert Space, and thus, has an internal product denoted as:  $\langle \cdot, \cdot \rangle$ , and usually defined as:

$$\langle f(x, t), g(x, t) \rangle = \int_0^T \int_{x_1}^{x_2} f(x, t)g(x, t)dxdt. \quad f, g \in L^2([x_1, x_2] \times [0, T], \mathbb{R})$$

Thus, the adjoint operator of  $L$  is obtained by finding  $L^*$  that holds:

$$\langle \lambda, L\psi \rangle = \langle L^*\lambda, \psi \rangle$$

Then, we need to solve:

$$\langle \lambda, L\psi \rangle = \int_0^T \int_{x_1}^{x_2} (\psi_t - d\psi_{xx} - r\psi + u\psi) \lambda dxdt.$$

It is important to notice that  $\lambda$  has not been defined, and its properties are yet to be explicitly stated. This will help us to build the operator  $L^*$ . We will analyze this term by term. Using integration by parts, we obtain:

$$\int_0^T \int_{x_1}^{x_2} \psi_t \lambda dxdt = \int_{x_1}^{x_2} \psi(x, T)\lambda(x, T) - \psi(x, 0)\lambda(x, 0)dx - \int_0^T \int_{x_1}^{x_2} \psi \lambda_t dxdt,$$

In this case, we have  $\psi(x, 0) = 0$ , and we can give  $\lambda$  the property  $\lambda(x, T) = 0$ , thus, we obtain:

$$\int_0^T \int_{x_1}^{x_2} \psi_t \lambda dxdt = - \int_0^T \int_{x_1}^{x_2} \psi \lambda_t dxdt.$$

For our next term, again using integration by parts, we obtain:

$$\begin{aligned}
&\int_0^T \int_{x_1}^{x_2} \psi_{xx} \lambda dxdt = \\
&\int_0^T \psi_x(x_2, t)\lambda(x_2, t) - \psi_x(x_1, t)\lambda(x_1, t)dt - \int_0^T \psi(x_2, t)\lambda_x(x_2, t) - \psi(x_1, t)\lambda_x(x_1, t)dt + \int_0^T \int_{x_1}^{x_2} \psi \lambda_{xx} dxdt.
\end{aligned}$$

Our border conditions for  $\psi$  give us that  $\psi(x_1, t) = \psi(x_2, t) = 0$ , and we can give  $\lambda$  the property:  $\lambda(x_1, t) = \lambda(x_2, t) = 0$ , thus obtaining:

$$\int_0^T \int_{x_1}^{x_2} \psi_{xx} \lambda dx dt = \int_0^T \int_{x_1}^{x_2} \psi \lambda_{xx} dx dt.$$

Finally, we add all these results together, and we get:

$$\langle \lambda, L\psi \rangle = \int_0^T \int_{x_1}^{x_2} (-\lambda_t - d\psi_{xx} + r\lambda - u\lambda) \psi dx dt. = \langle L^* \lambda, \psi \rangle$$

In this way, we have obtained our operator  $L^*$ . Now, we will assume we have an optimal control  $u^*$  for our problem. In that case, we will define our *adjoint equation* for our problem which corresponds to solve the following PDE, where  $u^*$  is our optimal control, and  $N^*$  is the optimal trajectory associated with the optimal control.

$$\begin{aligned} L^* \lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - u^* \lambda = \gamma N^* + cu^{*2} \\ \lambda(x_1, t) &= \lambda(x_2, t) = 0 \\ \lambda(x, T) &= 0 \end{aligned} \tag{5}$$

The right side of the equality in (5) is obtained by taking the derivative with respect to  $N$  in our objective functional (3), and evaluating in both the optimal control and trajectory.

Thus, we have obtained our adjoint system, which will aid us to solve the optimal control problem. Given that we are considering a minimization problem, assuming that we have an optimal control  $u^*$ , then our optimal control must have the following property:

$$\lim_{\epsilon \rightarrow 0} \frac{J[u_\epsilon^*] - J[u^*]}{\epsilon} \geq 0,$$

next, we will introduce the limit inside our integral formulation (we can do this using some version of the Dominated Convergence Theorem of Lebesgue, given that all the functions involved are continuous or piecewise continuous). Then:

$$0 \leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{\gamma}{2\epsilon} \left( (N_\epsilon^{*2} - N^{*2}) + \frac{c}{\epsilon} (u_\epsilon^{*2} N_\epsilon^* - u^{*2} N^*) \right) dx dt$$

We will consider each term in the integral. First, notice that:

$$\lim_{\epsilon \rightarrow 0} \frac{\gamma}{2\epsilon} (N_\epsilon^{*2} - N^{*2}) = \lim_{\epsilon \rightarrow 0} \frac{\gamma}{2\epsilon} (N_\epsilon^* - N^*) (N_\epsilon^* + N^*) = \gamma \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (N_\epsilon^* - N^*) \lim_{\epsilon \rightarrow 0} \frac{1}{2} (N_\epsilon^* + N^*)$$

Remember that:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (N_\epsilon^* - N^*) = \psi(N^*, u^*),$$

$$\lim_{\epsilon \rightarrow 0} N_\epsilon^* = N^*,$$

so, we get the following result:

$$\lim_{\epsilon \rightarrow 0} \frac{\gamma}{2\epsilon} (N_\epsilon^{*2} - N^{*2}) = \psi(N^*, u^*)\gamma N^*$$

For the next term of the integral, remember that:

$$u_\epsilon^* = u^* + l\epsilon,$$

then we have:

$$\begin{aligned} c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u_\epsilon^{*2} N_\epsilon^* - u^{*2} N^*) &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} ((u^* + l\epsilon)^2 N_\epsilon^* - u^{*2} N^*) = c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u^{*2} N_\epsilon^* + 2l\epsilon u^* + l^2 \epsilon^2 - u^{*2} N^*) \\ &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u^{*2} N_\epsilon^* - u^{*2} N^*) + \lim_{\epsilon \rightarrow 0} l^2 \epsilon + \lim_{\epsilon \rightarrow 0} 2cl u^* N_\epsilon^* \\ &= cu^{*2} \psi(N^*, u^*) + 2clu^* N^* \end{aligned}$$

Thus, returning to our integral, we have the following expression:

$$\begin{aligned} 0 &\leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{\gamma}{2\epsilon} ((N_\epsilon^{*2} - N^{*2}) + \frac{c}{\epsilon} (u_\epsilon^{*2} N_\epsilon^* - u^{*2} N^*)) dxdt \\ &= \int_0^T \int_{x_1}^{x_2} \psi(N^*, u^*)\gamma N^* + cu^{*2} \psi(N^*, u^*) + 2clu^* dxdt \\ &= \int_0^T \int_{x_1}^{x_2} (\gamma N^* + cu^{*2})\psi + 2clN^* u^* dxdt \end{aligned}$$

now, remember that the adjoint equation (5) has the following definition:

$$L^* \lambda = \gamma N^* + cu^{*2},$$

so, we have:

$$\begin{aligned} 0 &\leq \int_0^T \int_{x_1}^{x_2} L^* \lambda \psi + 2clN^* u^* dxdt \\ &= \int_0^T \int_{x_1}^{x_2} \lambda L \psi + 2clN^* u^* dxdt \\ &= \int_0^T \int_{x_1}^{x_2} 2clN^* u^* - lN^* \lambda dxdt \\ &= \int_0^T \int_{x_1}^{x_2} lN^* (2cu^* - \lambda) dxdt \end{aligned}$$

Clearly  $N^* \geq 0$ , and  $l$  is a perturbation function which may have any sign, because it represents perturbation in both directions from the optimal control. Thus, we can deduce that:

$$u^*(t, x) = \frac{\lambda(t, x)}{2c}.$$

What this indicates us is that our optimal control for this problem is dependent on the solution of the adjoint equation. We must also take in account the limits for our optimal control, and we obtain the final formula:

$$u^*(t, x) = \max\{0, \min\{\frac{\lambda(t, x)}{2c}, u_{max}\}\} \quad (6)$$

This is the optimal control obtained in [5]. We will use this characterization of the optimal control as a support for the rest of the mathematical results presented in this work.

### 3.2 Case 2: General Cost Function Analysis. Linear Growth Function

Our first aim, will be an attempt to generalize this result for different types of objective functions. We saw that in the reduction of:

$$\lim_{\epsilon \rightarrow 0} \frac{J[u_\epsilon^*] - J[u^*]}{\epsilon},$$

we reduced the formula to introduce  $L^*\lambda$  which allowed the characterization of our optimal control in terms of  $\lambda$ . Let's consider the same problem, with the same dynamic as described in (2), but with different objective functional:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P_k(N(x, t)) + cu^2(x, t)N(x, t) dx dt \quad (7)$$

$P_k(N)$  is a polynomial of degree  $k \in \mathbb{N}$ . Given that this a minimization problem, we may assume that the coefficients of  $P_k$  are all positive, such that when  $N$  is positive, the function behaves as a convex function, although this is not strictly necessary for the following analysis.

Given that  $P$  is a polynomial, its derivative with respect to  $N$  exists and is continuous with respect to  $N$ , and to represent it, we will use the notation  $P'_k(N)$ . So, following the same steps done in the preceding section, we can deduce that the adjoint equation associated with this (7) is:

$$\begin{aligned}
L^* \lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - u^* \lambda = P'_k(N^*) + cu^{*2} \\
\lambda(x_1, t) &= \lambda(x_2, t) = 0 \\
\lambda(x, T) &= 0
\end{aligned} \tag{8}$$

Now we'll proceed to solve:

$$0 \leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_k(N_\epsilon^*) - P_k(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*2} N_\epsilon^* - u^{*2} N^*) dx dt$$

We already solved the second limit, so we will focus in the first term. It is very intuitive that by taking the following limit:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_k(N_\epsilon^*) - P_k(N^*))$$

we should obtain something similar to the derivative of  $P'_k(N^*)$ , and we will see that is the case. First, it is clear that it will suffice to prove the result for just a polynomial with a single term such as:

$$P_k(N) = \frac{\gamma}{k} N^k,$$

because the rest of the terms of the polynomial will behave the same way. So, by solving for this term, we have:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_k(N_\epsilon^*) - P_k(N^*)) = \lim_{\epsilon \rightarrow 0} \frac{\gamma}{\epsilon k} (N_\epsilon^{*k} - N^{*k})$$

In this case, we can use the following algebraic identity, valid for any  $n \in \mathbb{N}$ :

$$a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \dots + a^2b^{n-2} + a^1b^{n-1} + b^n) = (a-b) \sum_{j=0}^n a^{n-j} b^j.$$

Thus, replacing in our formula we get:

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \frac{\gamma}{\epsilon k} (N_\epsilon^{*k} - N^{*k}) &= \frac{\gamma}{k} \lim_{\epsilon \rightarrow 0} \left( \frac{N_\epsilon^* - N^*}{\epsilon} \right) \sum_{j=0}^{k-1} N_\epsilon^{*n-j} N^{*j} \\
&= \psi(N^*, u^*) \gamma N^{k-1} \\
&= \psi P'_k(N^*)
\end{aligned}$$

Thus, we obtain the desired result, which, as we saw, in the previous section, allows the replacement with the formula for  $\lambda$  as we saw in the previous section, given that we obtain:

$$\begin{aligned}
0 &\leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_k(N_\epsilon^*) - P_k(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*2} N_\epsilon^* - u^{*2} N^*) dxdt \\
&= \int_0^T \int_{x_1}^{x_2} (P_k'(N^*) + cu^{*2})\psi + 2clN^*u^* dxdt \\
&= \int_0^T \int_{x_1}^{x_2} \psi L^* \lambda \psi + 2clN^*u^* dxdt \\
&= \int_0^T \int_{x_1}^{x_2} \lambda L\psi + 2clN^*u^* dxdt \\
&= \int_0^T \int_{x_1}^{x_2} 2clN^*u^* - lN^*\lambda dxdt \\
&= \int_0^T \int_{x_1}^{x_2} lN^*(2cu^* - \lambda) dxdt,
\end{aligned}$$

and, once again, this result allow us to characterize our optimal control as:

$$u^* = \max\{0, \min\{\frac{\lambda}{2c}, u_{max}\}\}. \quad (9)$$

We remark that, although this result seems to be the same as the one in (6), in fact is different, because the definitions for  $\lambda$  are different.

This result can be extended for functions which can be characterized as a sum of an infinite series of polynomials, such as the exponential function, and trigonometric and hyperbolic functions. A more immediate application for this result is that it easily allow for the addition of elements for cost and benefit analysis in our objective functional. For example, lets consider the following functional:

$$M[u] = \int_0^T \int_{x_1}^{x_2} b(\hat{N}(x, t) - N(x, t))dxdt,$$

where  $\hat{N}$  we will say is the trajectory of our model without taking in account any control measure, and  $N$  is the controlled model, and  $b$  can be considered some benefit factor. This functional can be taken as representing the benefit which our control is generating, for example, given the diminution of beaver population, this may represent a measure of ecological recovery, and it can be added, for example, to the analysis presented in [5], by switching the objective functional to:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} b(N(x, t) - \hat{N}(x, t)) + \frac{1}{2}\gamma N^2(x, t) + cu^2(x, t)N(x, t)dxdt, \quad (10)$$

and, we can construct the adjoint equation which characterizes the optimal control using the same methodology as described before.

Then, we already have changed the first term on our objective function. Now we will change the second term, and see how can we get some type of similar generalization for cost term which depends on the control. Consider the following objective functional:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P(N(x, t)) + cu^k(x, t)N(x, t) dx dt, \quad (11)$$

where,  $P$  is a polynomial of  $N$  of some degree, the same as described earlier, and we will require that  $k \in \mathbb{N}$ ,  $k \geq 2$ . Now, as before, we see that our adjoint equation associated with this functional is:

$$\begin{aligned} L^* \lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - u^* \lambda = P'(N^*) + cu^{*k} \\ \lambda(x_1, t) &= \lambda(x_2, t) = 0 \\ \lambda(x, T) &= 0, \end{aligned} \quad (12)$$

and as before, we'll take the limit:

$$\lim_{\epsilon \rightarrow 0} \frac{J[u_\epsilon^*] - J[u^*]}{\epsilon},$$

and we will try to find a characterization of the optimal control for this case in the same way as we did before. So, we will take on the computation of:

$$0 \leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P(N_\epsilon^*) - P(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*k} N_\epsilon^* - u^{*k} N^*) dx dt,$$

focusing in just the second term, given that we have already solve for the first term, and having in mind that:

$$u_\epsilon^* = u^* + l\epsilon,$$

thus, we have:

$$\begin{aligned} c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u_\epsilon^{*k} N_\epsilon^* - u^{*k} N^*) &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} ((u^* + l\epsilon)^k N_\epsilon^* - u^{*k} N^*) = c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( N_\epsilon^* \sum_{j=0}^k \binom{k}{j} u^{*k-j} (l\epsilon)^j - u^{*k} N^* \right) \\ &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u^{*k} N_\epsilon^* - u^{*k} N^*) + \lim_{\epsilon \rightarrow 0} \epsilon \sum_{j=2}^k \binom{k}{j} u^{*k-j} l^j \epsilon^{j-1} + \lim_{\epsilon \rightarrow 0} kcu^{*k-1} l N_\epsilon^* \\ &= cu^{*k} \psi(N^*, u^*) + kcu^{*k-1} l N^*, \end{aligned}$$

and, again, solving as before the whole expression, we get:

$$\begin{aligned}
0 &\leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P(N_\epsilon^*) - P(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*k} N_\epsilon^* - u^{*k} N^*) dxdt \\
&= \int_0^T \int_{x_1}^{x_2} (P'_k(N^*) + cu^{*k})\psi + kclN^* u^{*k-1} dxdt \\
&= \int_0^T \int_{x_1}^{x_2} \psi L^* \lambda \psi + kclN^* u^{*k-1} dxdt \\
&= \int_0^T \int_{x_1}^{x_2} \lambda L \psi + kclN^* u^{*k-1} dxdt \\
&= \int_0^T \int_{x_1}^{x_2} kclN^* u^{*k-1} - lN^* \lambda dxdt \\
&= \int_0^T \int_{x_1}^{x_2} lN^* (kc u^{*k-1} - \lambda) dxdt,
\end{aligned}$$

finally, in this case we may characterize our optimal control as:

$$u^* = \max\{0, \min\{\sqrt[k-1]{\frac{\lambda}{kc}}, u_{max}\}\}. \quad (13)$$

This characterization shows us why we selected values of  $k \geq 2$ , given that for  $k = 1$ , the result is not applicable. In this case, the situation we have is a linear dependence of the control in the objective function, which may indicate that the use of a Bang Bang control may be required.

We selected to focus just in the case  $u^k$ , because as we see, in that case we can easily characterize the control in terms of the adjoint equation. Of course, in principle we may select, for example, a polynomial for  $u$ , but it's clear that such case would difficult the selection of an optimal control, given the existence of multiple solutions for such polynomial.

Next, we will extend this result one more time. This time our objective functional will be:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P(N(x, t)) + cu^k(x, t)Q(N(x, t)) dxdt, \quad (14)$$

where  $P$  is a polynomial of some degree, the same way as we defined it before, and  $Q$  is a polynomial of any degree. We denote  $Q$  derivative with respect to  $N$  as  $Q'(N)$ . So, we proceed as usual: First, we define our adjoint function:

$$\begin{aligned}
L^* \lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - u^* N^* \lambda = P'(N^*) + cu^{*k} Q'(N^*) \\
\lambda(x_1, t) &= \lambda(x_2, t) = 0 \\
\lambda(x, T) &= 0
\end{aligned} \quad (15)$$

Then, as before, we solve for our functional:



$$\lim_{\epsilon \rightarrow 0} \frac{J[u_\epsilon^*] - J[u^*]}{\epsilon},$$

and apply the property:

$$0 \leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P(N_\epsilon^*) - P(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*k} Q(N_\epsilon^*) - u^{*k} Q(N^*)) dxdt,$$

we already solve for the first term, now we solve for the second term, similarly as we did before.

We have:

$$\begin{aligned} c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u_\epsilon^{*k} Q(N_\epsilon^*) - u^{*k} Q(N^*)) &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} ((u^* + l\epsilon)^k Q(N_\epsilon^*) - u^{*k} Q(N^*)) \\ &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( Q(N_\epsilon^*) \sum_{j=0}^k \binom{k}{j} u^{*k-j} (l\epsilon)^j - u^{*k} Q(N^*) \right) \\ &= c \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u^{*k} Q(N_\epsilon^*) - u^{*k} Q(N^*)) + \lim_{\epsilon \rightarrow 0} \epsilon \sum_{j=2}^k \binom{k}{j} u^{*k-j} l^j \epsilon^{j-1} Q(N_\epsilon^*) + \lim_{\epsilon \rightarrow 0} kc u^{*k-1} l Q(N_\epsilon^*) \\ &= cu^{*k} \psi(N^*, u^*) Q'(N^*) + kc u^{*k-1} l Q(N^*). \end{aligned}$$

The first element in the sum is obtained by the same argument we solved for  $P$  before.

Thus, we solve the rest of our problem, having in mind that  $Q(N) = N \frac{Q(N)}{N}$ . Then:

$$\begin{aligned} 0 &\leq \int_0^T \int_{x_1}^{x_2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P(N_\epsilon^*) - P(N^*)) + \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon} (u_\epsilon^{*k} Q(N_\epsilon^*) - u^{*k} Q(N^*)) dxdt \\ &= \int_0^T \int_{x_1}^{x_2} (P'_k(N^*) + cu^{*k} Q'(N^*)) \psi + kc Q(N^*) l u^{*k-1} dxdt \\ &= \int_0^T \int_{x_1}^{x_2} \psi L^* \lambda \psi + kc Q(N^*) l u^{*k-1} dxdt \\ &= \int_0^T \int_{x_1}^{x_2} \lambda L \psi + kc Q(N^*) l u^{*k-1} dxdt \\ &= \int_0^T \int_{x_1}^{x_2} kcl \frac{Q(N^*) N^*}{N^*} u^{*k-1} - l N^* \lambda dxdt \\ &= \int_0^T \int_{x_1}^{x_2} l N^* \left( \frac{Q(N^*)}{N^*} kc u^{*k-1} - \lambda \right) dxdt, \end{aligned}$$

finally, in this case we characterize our optimal control as:

$$u^* = \max\{0, \min\{k^{-1} \sqrt{\frac{\lambda N^*}{Q(N^*) kc}}, u_{max}\}\} \quad (16)$$

which, as before, the result may be generalized for any function which can be represented as an infinite polynomial. Considering this is a minimization problem, we may need to add further conditions for  $Q$ , such as to require it to be convex for positive values of  $N$ .

To close this section, we will notice how this general scheme we described for this part can be used for the cost scheme we described in the previous section, given that  $P(N)$ , for

example, can be considered a polynomial with costs associated with Ecological Costs of the beaver infestation, and  $Q(N)u^k$  can be considered the modeling of Trapping Costs of the beaver infestation. In the next section we will notice the usefulness of this result.

### 3.3 Case 3. General Cost Function. Restricted Space. Linear Growth Function

After dealing with different alternatives for the objective function, now we will restrict the control space. For example, given our interval  $I = [x_1, x_2]$ , we may want to restrict the control in an interval  $S$  such that  $S \subset N$ , forcing in the set  $I - S$  the control to be 0.

Defined in such a way, our PDE equation now would be:

$$\begin{aligned} N_t &= dN_{xx} + rN - \mathbf{1}_S uN \\ N(x_1, t) &= N(x_2, t) = 0 \\ N(x, 0) &= g(x), \end{aligned} \tag{17}$$

and thus, our objective functional, considered in the general form we computed earlier, would be:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P(N(x, t) + c\mathbf{1}_S u^k(x, t)Q(N(x, t))) dx dt, \tag{18}$$

where,  $P$  and  $Q$  are polynomials of  $N$ . For this case, we will make an analysis restricted in our subset  $S$ . For sake of simplicity, we will assume that  $S$  is a single fixed interval of the form  $S = [x_3, x_4]$ , and inside that interval, we will define the perturbed control:

$$u_\epsilon = u + l\epsilon,$$

where  $l$  is a perturbation function. Outside of  $S$ , the control can not be perturbed, because we are forcing  $u = 0$ , and thus, for this analysis we consider  $l = 0$ . Thus, our sensitivity function is defined as:

$$\psi(x, t) = \lim_{\epsilon \rightarrow 0} \frac{N_\epsilon(x, t) - N(x, t)}{\epsilon},$$

where  $N_\epsilon(x, t)$ , is the trajectory associated with the perturbed control. Clearly, outside of  $S$ , we will have that  $\psi = 0$ , and for that reason, we will force our border conditions to be 0 also for the perturbation equation, to avoid discontinuities. This point should be explored further,

because although this analysis obtains a control which is a critical point of our functional by these conditions, the optimality of this control may be in question by forcing the variation of the optimal control in the border to be zero. This problem should be addressed in further research, although, in the results sections it will be noticed that controls defined with these conditions still are useful for the control problem at hand.

Then, we can build the following dynamic of  $\psi$ , the same way as we did it in our first case, and we describe it by:

$$\begin{aligned}
\psi_t - d\psi_{xx} - r\psi + u\psi &= -lN \\
\psi(x_3, t) = \psi(x_4, t) &= 0 \\
\psi(x, t) = 0 \quad x \in I - S \\
\psi(x, 0) &= 0.
\end{aligned} \tag{19}$$

We can repeat the analysis done in the previous section for computing the adjoint equation. The analysis will be the same, just replacing the limits of the integrals with  $[x_3, x_4]$ , instead of  $[X_1, x_2]$ . Thus, we obtain for the adjoint PDE:

$$\begin{aligned}
L^*\lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - u^*\lambda = P'(N^*) + cQ'(N^*)u^{*k} \\
\lambda(x_3, t) = \lambda(x_4, t) &= 0 \\
\lambda(x, t) = 0 \quad x \in I - S \\
\lambda(x, T) &= 0,
\end{aligned} \tag{20}$$

And, thus we compute:

$$\lim_{\epsilon \rightarrow 0} \frac{J[u_\epsilon^*] - J[u^*]}{\epsilon},$$

and the analysis will be the same as before, with the special consideration that the limits of the integrals in space are between  $[x_3, x_4]$  rather than in  $[x_1, x_2]$ . Finally, our control in this restricted case will be characterized as:

$$u^* = \begin{cases} \max\{0, \min\{k^{-1}\sqrt{\frac{\lambda}{kcR(N^*)}}, u_{max}\}\} & \text{if } x \in S \\ 0 & \text{if } x \in I - S \end{cases} \tag{21}$$

Clearly, restricting the control to a particular zone seems to be useless at first glance, given that this control only can be suboptimal compared to the control (13), which is applied in the

whole interval  $I$ . But, consider the following question: what if our upper bound  $u_{max}$  can be larger in a restricted sector. In such a case, obtaining a better control policy would again be in question, given that the characterization would clearly change.

Such a case has an application, for example making comparison between different types of strategies of control: What would be better, to apply the control in the whole space, but with a lesser maximum ratio of trapping, or the application of control in a restricted area, but with greater rate of trapping. This problem easily can generate further problems, such as applying mixed strategies, such as, to apply total control until a switching time  $T_s$ , and then focused control for the rest of the time. Or perhaps to apply a control in a zone  $S$  until a switching time  $T_s$ , and then focus the control strategy in  $I - S$ . These considerations open even more questions such as the selection of the optimal switching time in applying a particular strategy.

In this work, we suggest the following criteria for selecting the new value of  $u_{max}$  for the restricted control problem. Given an interval  $I = [x_1, x_2]$  of length  $L$ , and another interval  $S = [x_3, x_4]$  of length  $K$ , such that  $S \subset I$ .  $u_{max}$  is the maximum trapping rate for the control in the whole interval. As we said in the previous section,  $u_{max} \in [0, 1]$ , so, in the restricted interval,  $u_{Kmax}$  can be defined:

$$u_{Kmax} = \frac{L - K}{L} + u_{max} \frac{K}{L}.$$

As we see, if  $K = L$ , in which  $I = S$ , then,  $u_{Kmax} = u_{max}$ , and if  $K = 0$  (control applied in a single point in such case), then  $u_{Kmax} = 0$ , an impossible case. Other criteria may be considered, but for the present study, this formula seems an adequate modeling tool for representing increasing trapping effort in a more restricted zone, as it will be developed later on.

### 3.4 Case 4: General Cost Function, Multiple Controls, Linear Growth Function

As we discussed in the previous section, in general we consider our main control policy to be trapping techniques, which is the control policy suggested in works such as [5] [1]. But, in case we may consider different policies, for different zones, our previous result may be considered very useful.

For example, consider we are restricting trapping into some interval  $S_1$ , and in another sector  $S_2$  we are considering a hunting policy encouraged by the government, For sake of

simplicity, we will assume that  $S_1 \cap S_2 = \emptyset$ , which represents, for example, that the implementation of a trapping policy will be in zones in which there is no hunting. By setting  $u_1$  as our trapping ratio, and defining  $u_2$  as our hunting ratio, then, following our general schema, our objective functional would be:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P(N(x,t) + \mathbb{1}_{S_1} u_1^k(x,t) Q_1(N(x,t)) + \mathbb{1}_{S_2} u_2^j(x,t) Q_2(N(x,t))) dx dt, \quad (22)$$

where  $P$ ,  $Q_1$  and  $Q_2$  are polynomials with the characteristics we required in the previous section. The dynamic associated with the beaver population in this case will be described by:

$$\begin{aligned} N_t &= dN_{xx} + rN - \mathbb{1}_S u_1 N - \mathbb{1}_S u_2 N \\ N(x_1, t) &= N(x_2, t) = 0 \\ N(x, 0) &= g(x), \end{aligned} \quad (23)$$

and, by solving for each control in its own space, clearly we'll get that our control for this case will be characterized as:

$$u_1^* = \begin{cases} \max\{0, \min\{\frac{\lambda_1 N^*}{k Q_1(N^*)}, u_{1max}\}\} & \text{if } x \in S_1 \\ 0 & \text{if } x \in I - S_1 \end{cases} \quad (24)$$

$$u_2^* = \begin{cases} \max\{0, \min\{\frac{\lambda_2 N^*}{j Q_2(N^*)}, u_{2max}\}\} & \text{if } x \in S_2 \\ 0 & \text{if } x \in I - S_2 \end{cases}, \quad (25)$$

where  $\lambda_1$  and  $\lambda_2$  are the respective adjoint equations of the restricted problems associated with  $u_1$  and  $u_2$  respectively.

For controls which are active in overlapped domains, the question of characterizing such a control becomes harder, and its left as an open question.

### 3.5 Case 5: General Cost Function. Logistic Growth Function

In previous sections, we defined our original problem with the PDE (2):

$$\begin{aligned} N_t &= dN_{xx} + rN - uN \\ N(x_1, t) &= N(x_2, t) = 0 \\ N(x, 0) &= g(x) \end{aligned}$$

which represents a diffusion of the population at an exponential growth rate. All the previous calculation was done with this growth function for sake of simplicity, but in general, as is modeled in [1], [5], [19], a logistic growth rate is used, this is, changing our model to:

$$\begin{aligned}
N_t &= dN_{xx} + rN \left( 1 - \frac{N}{N_{max}} \right) - uN \\
N(x_1, t) &= N(x_2, t) = 0 \\
N(x, 0) &= g(x)
\end{aligned} \tag{26}$$

where  $N_{max}$  represents the carrying capacity of the environment for the beaver population. We will set our optimization problem, then, in our general form:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} P(N(x, t) + cu^k(x, t)Q(N(x, t))) dx dt, \tag{27}$$

Where, let's remember,  $P$  is any polynomial, and  $Q$  is a polynomial of the form:  $Q(N) = NR(N)$ , where  $R$  is a polynomial of a lesser degree. Mathematically, our calculations developed in the previous sections are not particularly different for the ones needed to deduce the systems for a different growth rate. The main difference will be in the computation of the sensitivity function, and thus, the adjoint equation, which in turn characterizes a different optimal control. Using the same definitions as before, with logistical growth rate, we obtain the following sensitivity PDE:

$$\begin{aligned}
\psi_t - d\psi_{xx} - r\psi + \frac{2rN}{N_{max}}\psi + u\psi &= -lN \\
\psi(x_1, t) = \psi(x_2, t) &= 0 \\
\psi(x, 0) &= 0
\end{aligned} \tag{28}$$

and, thus, we obtain the following adjoint equation:

$$\begin{aligned}
L^* \lambda &= -\lambda_t - d\lambda_{xx} + r\lambda - \frac{2rN^*}{N_{max}}\lambda - u^* \lambda = P'(N^*) + cu^{*k}Q'(N^*) \\
\lambda(x_1, t) &= \lambda(x_2, t) = 0 \\
\lambda(x, T) &= 0.
\end{aligned} \tag{29}$$

Which characterizes the following optimal control:

$$u^* = \max\{0, \min\{k^{-1} \sqrt{\frac{\lambda N^*}{kcQ(N^*)}}, u_{max}\}\}. \tag{30}$$

we stress again, although it seems to be the same control as the one we got (16), it is a different control, because it is associated with a different adjoint equation and operator.

We close this section at this point, with the problem defined in the way as will be studied in the following section, and with multiple mathematical tools developed for dealing with the control problem, which will be applied for our design of trapping strategies for the beaver infestation.

## 4 Results

After having developed the mathematical results for dealing with our control problem, in the present section, we will perform numerical tests, using modeling of different optimization strategies. Our main tool for this will be the Backward-Forward Algorithm, which, described in a general way, consists in setting up initial control (for example, 0, no control situation), and then, using that control, compute the trajectory  $N$  associated with such control. Then, using the trajectory, compute the associated adjoint equation numerally also, finally, using the adjoint equation and the trajectory, a new control is created using the formulas we obtained in our previous analysis for the characterization of the optimal controls, which is then fed into the algorithm, creating a new trajectory and adjoint function, thus repeating the process, until some convergence ratio is reached. For further reference on the Backward-Forward Algorithm, refer to [5] [9] (This last reference also contains discussion for convergence conditions), and also there is a pseudo-language description in Appendix 7.1. For our work, we adjusted the algorithm to include spatial considerations, as was suggested by [5]. The implementation of this was done in Matlab.

For our tests, we used the information available in [19] and [17] for modeling the dynamics of beaver growth and rates of expansion. In Table 4, we can see some of the values obtained for describing the dynamics of beaver population growth.

Regarding the dynamics of beaver spreading, we use the values of the discussion assessed in [19], we mentioned earlier. Diffusion rates are not constant, and in [19] is suggested there's probably a factor of resource depletion driving the expansion. Such an analysis would require to switch the diffusion rate, which is complicated for the analysis and computing the optimal control, thus, it will be left as an open problem. The diffusion rates suggested in [19] will be used, then, taking in account the geographical zones we're considering in each of our tests.

Area	Zone	Beaver Colonies/km	Water Courses (km)	Estimated number of beavers/colony	Estimated abundance of beavers
TDF	North	0.15	2,042.9	4	1,226
TDF	Central	0.64	1,711.2	5	5,746
TDF	South	1.91	3,630.6	5	34,672
NAV	-	1.1	3,634.3	5	19,187
Total					61,363

Table 4: Density and estimation of the abundance of beavers on Tierra del Fuego (TDF) and Navarino (NAV) in the year 2000 (Table obtained in [17])

Element	Value
Growth Function	Logistical
$N_{max}$	5.5 [17]
$r$	0.18 [19]
$d$	4.45 [17][19]
$\gamma$	701.79[1]
$c$	149.82[1]

Table 5: Parameters used for the equations and cost functions in the numerical tests based on the data of Navarino Island.

#### 4.1 Analysis using Quadratic Cost Function

In our first set of computational tests, we'll use the data of Isla Navarino. Isla Navarino is an island located at the south side of the main island of Tierra del Fuego. Belongs to Chile, and has a small population of 1677 habitants. Is one of the most remote zones of Chile. The island is full of small lakes and water courses, which serve as spots for beaver colonization. In [19], beaver carrying capacity in Navarino Island is given as 20,000, but in [16], the population is estimated at 19,987. Such considerations makes us assume that population of beaver in Navarino island should be now near its carrying capacity. The detail of the parameters used for modeling the population and diffusion dynamics is shown in Table 5 The diffusion rate we deduced it from taking the average value of the range of diffusion speed values estimated in [17]. For our space, we'll consider a 100 km zone and a time horizon of 10 years. First, we



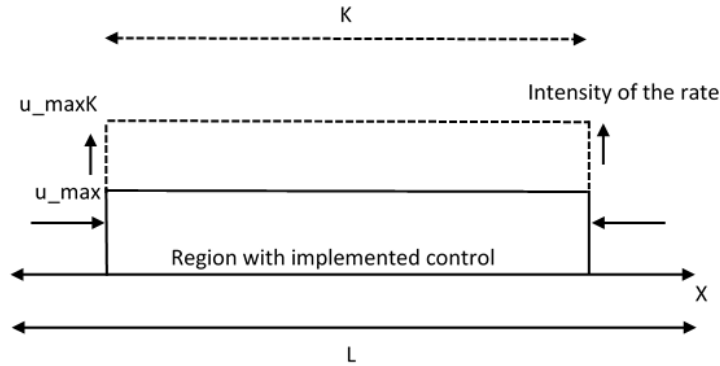


Figure 2: Scheme of different strategies of control considered. The space is being reduced from  $L$  to  $K$ , but a greater maximum culling rate  $u_{maxK}$  is obtained.

will test the policies using the objective function proposed in [1], and using the cost values suggested in that work (the values have been adjusted for inflation). Thus, our objective function is:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} \left[ \frac{1}{2} \gamma N^2(x, t) + cu^2(x, t)N(x, t) \right] dxdt$$

The values of the parameters are in table 5. These values are used just for sake of getting an initial reference, and then to compare the results with studies of cost estimations made for the beaver infestation problem in Tierra del Fuego. The damage parameter was estimated using local damage reports and beaver populations, and the cost parameter just considers wages costs [1]. Then, we'll setup our initial population for control. Given that currently, as we previously said, the beaver population is estimated to be near of its carrying capacity, any characterization of an initial distribution should be based on the assumptions of such studies, but we currently have not access to the study in which these data is estimated. Thus, we'll assume that given the density we have as a value, we'll setup an initial population of beavers in our water course by creating a random distribution of beavers centered in 5.5 and whose values don't go below 5 or over 6. This is considered a fair assumption, given the average sizes of beaver colonies, the data presented in [17] and [19], and also, the assumption is that the control is being applied in a densely populated area.

Now, we will setup the values for rates of beaver trapping. Beaver trapping is considered a lifestyle in countries such as the United States, and there exists a culture of trapping, specialized equipment and techniques that have been developed over generations. Such a

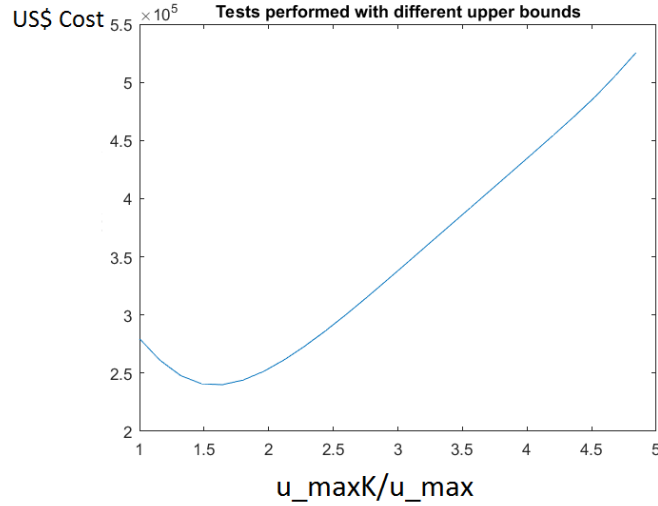


Figure 3: Cost performance of tests considered with different rates of maximum capture.

situation does not exist in Chile, in which the area affected is remote, scarcely populated, and without a native culture of trapping practices, thus, we start assuming initially a low maximum capture rate of  $u_{max} = 0.2$  along our area. Finally, our tests assumed that controlling in a restricted area of length  $K$ , will improve the maximum value available for the control by the formula:

$$u_{Kmax} = \frac{L - K}{L} + u_{max} \frac{K}{L},$$

where, in this case  $L = 100km$ . This represents a more concentrated control policy, leaving zones without trapping practices. In the experiments we performed, we started with areas of 4 kms from the center of the interval, and then we expanded it, by 2 kilometers on each side in each iteration, and adjusting the maximum value, and using the adjoint equation and characterizations obtained in the previous section. An scheme of this can be seen in figure 2.

Repeated tests using this criteria showed that our optimal control applied in the entire space is suboptimal compared with a more focused strategy. The relevant results in this case are shown in table 6.

We can see that the reduction of controlled space, with greater rates of capture, has a good effect for the costs in the parameters we have setup for the problem. A summary of all the results can be seen in Figure 3, and graphics of the populations can be seen in figure 4, where the differences between the characterizations are noticeable. In Figure 5 the final distribution of beaver population can be seen in the plane. Another point to notice is the

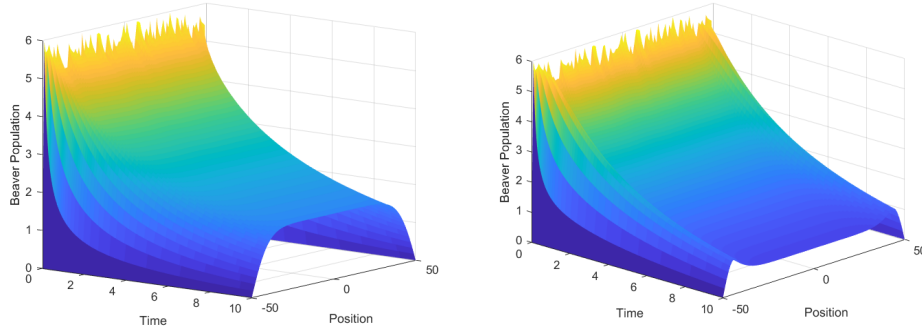


Figure 4: In the left, beaver population with control applied in the entire space. In the right, beaver population with control applied in 84% of space.

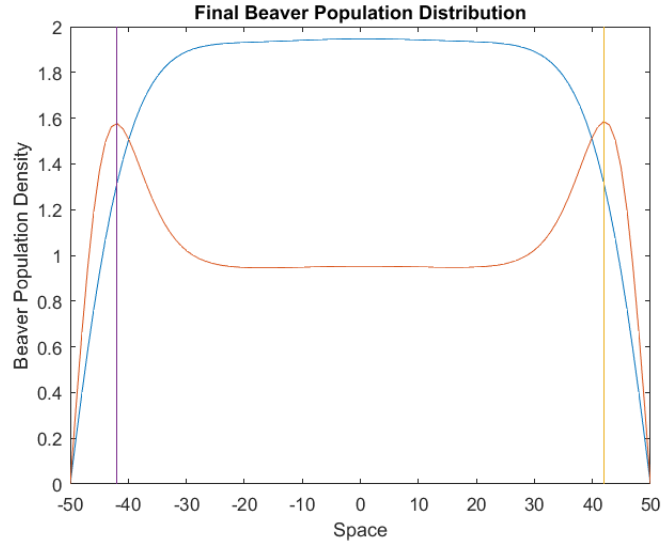


Figure 5: Distribution of Beaver Population at final time. Vertical lines represent the limits of control policy.

Pars.	$u_{max}$	Cost Base	Best $u_{maxK}$	% Sp.	Control Cost	Damage Costs	Total Cost
Normal	0.2	US\$3,080,750.13	0.328	84%	US\$28,018.05	US\$2,055,266.55	US\$ 2,083,284.60
High	0.2	US\$6,216,121.36	0.328	84%	US\$56,273.60	US\$4,155,251.54	US\$ 4,211,525.14

Table 6: Results for tests using the different parameter values suggested in [1]. % Sp. represents the space covered by the optimal strategy. 'Pars.' indicate which parameter estimation is being considered. Normal considers just the normal parameter values estimated in [1]. High considers the largest parameter values.

difference between ecological costs and trapping costs is noticeable with the data provided in [1]. For the best control policy, the results are shown in table 6.

These results suggest that with these cost considerations, the main component of the cost is driven by the damage costs rather than the trapping costs. Our next step in our testing was to repeat the analysis, this time using the cost parameters suggested in [1], but at their maximum estimated values:

- $\gamma = 1,405.63$ .
- $c = 299.87$ .

Repeating the analysis using these higher estimation, we obtained the same qualitative result: to control in a reduced space is more effective, and the results are shown in table 6.

We can conclude that, using these estimations, that the main cost of the strategy is driven by damage costs, and that reducing the control space can yield better cost results. These data were obtained in the design of a trapping strategy in the United States where, as we already mentioned, there exist a culture of trapping. An effective trapping strategy does not just consists in setting up traps, but to know the terrain and to understand the behavior of the species, which indicates the trapping costs can not just consider the costs for individual traps, there is also the skill and effective design of the trapping strategy.

In [11], the estimations of expenditure for the Chilean government for three years for training and supply of trappers in all of Tierra del Fuego was put on US\$759,643.80 (The figure has been adjusted by inflation). Considering that our previous analysis just controlled in a 100km sector, we could initially estimate the cost for control in all the waterways of Navarino Island (3600km, as seen in table 4), in about US\$2,025,849.6, just for control costs, and just for Navarino Island. Although this is a very rough estimation, it seems to indicate that, governments spent lower cost values of what should be expected for an effective control strategy in the United States. Our initial assessment of control policies in Navarino Island then seem to indicate that what the Chilean government spent 10 years ago in all of Patagonia. About the damage costs, we will discuss them in the next section, and analyze how well these damage costs fit with economic estimations.

All the previous analysis was done under the assumption of  $u_{max} = 0.2$  when our control policy is applied in the whole space. This assumption, of course, can be criticized, given the announcements of governments to deal throughly with the infestation. Then, we ran tests for ran tests for  $u_{max} = 0.5$  and  $u_{max} = 0.8$ , and results are given in table 7.

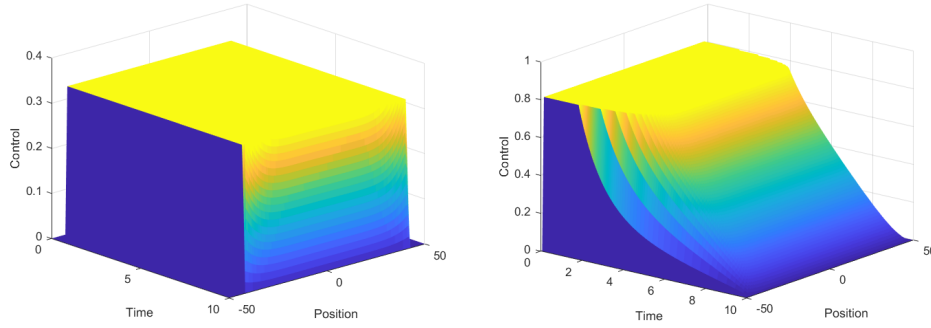


Figure 6: In the left, best control when  $u_{maxK}$  is 0.32, applied at 82% of the space. In the right, optimal control when  $u_{maxK}$  is 0.82, applied at 96% of the space.

Pars.	$u_{max}$	Cost Base	Best $u_{maxK}$	% Sp.	Control Cost	Damage Costs	Total Cost
High	0.2	US\$6,216,121.36	0.328	84%	US\$56,273.60	US\$4,155,251.54	US\$ 4,211,525.14
High	0.5	US\$2,389,619.12	0.54	92%	US\$96,283.37	US\$2,162,683.45	US\$ 2,258,966.83
High	0.8	US\$1,508,036.42	0.82	96%	US\$139,073.81	US\$1,359,228.48	US\$ 1,498,302.29

Table 7: Results for tests using the different higher values suggested in [1], but considering different base rates. % Sp. represents the space covered by the optimal strategy. 'Pars.' indicate which parameter estimation is being considered.

About these results, we can see that, our strategy of reducing the controlled space but increasing the maximum culling ratio, still deliver better results than controlling in the entire space, but the largest we take the maximum culling rate, the lower the improvement gets. Our optimal controls obtained also have different characteristics, as can be seen in figure 6. In both cases, a constant culling strategy is constantly applied, but with  $u_{maxK} = 0.82$ , the application of the maximum culling strategy is applied less than when  $u_{maxK} = 0.328$ , in which, the maximum strategy is applied almost the entire time. In a way, this result is expected, given that we are applying a better culling rate, and thus, by reducing drastically the number of beavers, the constant application of the maximum culling rate is not so necessary in  $u_{maxK} = 0.82$ , than with  $u_{maxK} = 0.328$ . It is noticeable, also, the reduction of ecological costs, which make the estimated costs way lower when highest rates are considered. Also, the estimated control costs are greatly increased. The results can be seen in table 7

Let's keep in mind that these numbers are just considered for a 100km zone in Navarino Island. A rough estimation of the control costs of covering the whole island for these strategies

are US\$3,466,201.32 for 54% coverage and US\$5,006,657.16 for 82% coverage. Comparing this data with the expenditure of the Chilean government we covered earlier, we can reconfirm our previous argument about the low investment the government has done. This rough estimate seem to be somewhat in line with the costs estimated in [11] for the estimated costs for full eradication. (For reference sake, in [11], the total cost of eradication is estimated in total in US\$38,550,446.15, data adjusted for inflation)

## 4.2 Alternative considerations for the Cost Functional

In our previous discussion, we used the values, adjusted by inflation, presented in [1], for estimating costs and damage values associated to our problem. But there is much to criticize about using these values for cost estimates in the problem in Tierra del Fuego. The following considerations are to be taken for improving a cost estimation:

- Use of local costs values for the consideration of cost functions.
- Use of geographical local data for considering the cost function.

In the previous section we saw that with the assumptions given in [1] for trapping costs, we obtained similar results to the expenditure done by the Chilean government for a three-year plan for covering the whole area. Thus, although further adjustment is necessary, and possibly we are underestimating the costs we will not consider necessary to adjust the parameters of this function, although further work along the lines to estimate the trapping costs is considered necessary for a better analysis.

We will discuss now the cost function associated with damage costs. We saw that, using the damage costs proposed in [1], the damage costs were very high in comparison with the trapping costs. But that cost estimation was considered by measuring local damages, and is not taking in consideration the conditions and costs associated with the situation in Tierra del Fuego. Thus, we will mainly focus in making adjustments in the damage cost for this section.

A study from 2014 [18] makes a possible evaluation of the total costs of Native Forest along the lines of US\$ 7,535,020.30 (data adjusted for inflation) per year. Our previous rough estimate, just for Navarino Island, gave us an estimated cost of US\$ 149,589,055.44 for ten years, while applying a rather low control policy, and taking the higher damage values suggested by [1]. Then, it seems that our previous analysis overestimated the damage costs, although we must stress that the damage cost was estimated with the highest value estimated

Economic Group	C2	Average	ABC1
Beaver	0.5	1	1.5
Cost Associated	US\$55.34	US\$75.35	US\$118.6

Table 8: Estimation data obtained from [18].

in [1]. Repeating the analysis, this time taking the higher control costs, but the normal damage estimates presented in [1] gave us the following results:

- **100% of space with traps, 0.2 rate of capture** Final cost: US\$3,094,995.56
- **84% of space with traps, 0.32 rate of capture** Final cost: US\$2,101,104.36

And, the costs components of the best control policy are:

- **Best control: Trapping Costs** Final cost: US\$55,888.15
- **Best control: Damage Costs.** Final cost: US\$2,045,216.21

These results give us a rough estimate of US\$ 73,627,783.56 of damage, just for Navarino Island, which seems to be more adjusted for the figures considered in [18], although that estimation is for the whole area, thus even considering the low estimation done by [1] for the damages cost, we should consider it still to be an exaggerated figure.

This difference in cost estimations can be explained by taking in consideration that the zone of Tierra del Fuego has a low population, and mainly concentrated in urban zones (97% urban vs 3% rural). The data collected in [18] was considered by means of surveys of the local population and by taking in account the values of costs the subject were willing to pay for the problem, and the main considerations seemed to be in terms of ecologic preservation of the habitat rather than economic, which was the focus of the damage estimation in [1]. In [18] the total figure of US\$ 74,636,656.46 in ecological costs is presented, although we could not get the data of the study they referenced, about the time frame of this estimation, and area considered.

All the previous argument seem to suggest that the reference values provided by [1] for damage costs can not be applied to our problem, and new parameter cost must be estimated. In our study, we have provided mathematical tools for the use of any polynomial for the estimation of damages, thus we will provide a rough estimation using the data available in [18], and we will compare it with the results previously obtained.

First, in [18], the surveys estimated a yearly figure of US\$ 7,535,020.30. Estimating a Beaver population of 100,000 total in Chilean Tierra del Fuego, this presents a cost of US\$75.35 per beaver the Chilean public is willing to pay. Using this parameter, we estimated the following initial cost function:

$$E_1(N) = 75.35N.$$

This function is simple, although it may be considered somewhat conservative. These costs estimates are given by public opinion, thus, using additional data obtained in [18], as can be seen in table 8, from different socioeconomic strata, using Matlab we interpolated the following cost curves:

$$E_2(N) = 7.43N^2 + 67.9N$$

$$E_3(N) = 78.09N^3 - 187.80N^2 + 185.06N$$

Our polynomials  $E_2$  and  $E_3$  assume that the greater the beaver population gets, the perceived estimations of cost by the public will increase, thus the quadratic and cubic exponents. In general, for cost estimation, quadratic or linear estimates are considered, and we could not find references for cubic costs. Our assumption for proposing the cubic cost function is that, given that the costs estimated in [18] are driven by public perceptions and ecological sensitivity, a greater beaver problem can generate huge changes in the public perception of the problem, and thus of the costs the public is willing to pay for the solution of the problem. In a way, these costs functions represent how the public opinion may change if the beaver problem remains untreated and the ecological damage caused by the beavers grow out of hand. A graphic of these functions can be seen in 7.

Thus, our cost functional will be:

$$\min_{u \in U} \int_0^T \int_{x_1}^{x_2} E_j(N) + cu^2(x, t)N(x, t)dxdt$$

$$j = 1, 2, 3$$

- Trapping associated cost parameter:  $c = 299.87$ .

We reiterate that  $c$  is the higher estimated cost associated by trapping proposed in [1], which we considered an acceptable approximation given the cost values discussed in the previous section.



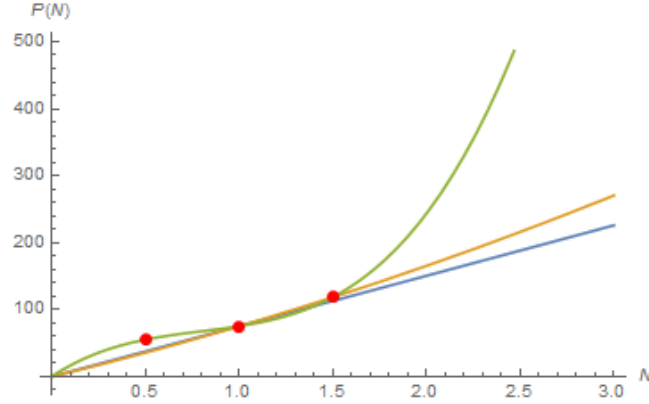


Figure 7: Different cost functions used. In blue is  $E_1(N)$ , in yellow is  $E_2(N)$ , in green is  $E_3(N)$ .

Cost F.	$u_{max}$	Cost Base	Best $u_{maxK}$	% Sp.	Control Cost	Damage Costs	Total Cost
$E_1$	0.2	US\$233,650.19	0.296	88%	US\$46,701.62	US\$163,925.29	US\$210,626.91
$E_2$	0.2	US\$280,784.66	0.328	84%	US\$51,778.72	US\$189,463.48	US\$241,242.20
$E_3$	0.2	US\$1,389,062.96	0.328	84%	US\$50,841.79	US\$905,211.98	US\$956,053.78

Table 9: Results for tests using the different cost functions constructed using [18] values. 'Cost F.' indicates the cost function used. % Sp. represents the space covered by the optimal strategy.

Thus, we ran the algorithms for our three cost functions, taking a low maximum culling rate (0.2). The results are shown in Table 9. Qualitatively, the results are similar as the obtained in the previous section. The cost results got lower, as expected.

The previous exercise seem to corroborate that, even by taking lower costs associated with damages, there seems to be an improvement in control policy by restricting the space controlled, but increasing the culling ratio. We must also notice that the function we estimated  $E_3$  is not convex in the domain we considered, thus questions may arise about the optimality of the control found, though the convergence of the Backward-Forward algorithm and the similitude with the qualitative behavior in the linear and quadratic cases seem to indicate the validity of the estimation.

Also we notice that our three functions delivered similar costs in terms of control policy, but our different ecological costs functions delivered quite different estimations for the problem. Computing again the rough estimation we done previously, this is, multiplying these cost estimates for 36, taking in account all the waterways of Navarino Island, the costs associated are:

- $E_1$  Total cost: US\$5,901,310.44
- $E_2$  Total cost: US\$6,820,685.28
- $E_3$  Total cost: US\$32,587,631.28

Let's remember that these estimated costs are associated for a ten year policy, and just in Navarino Island, and seems to be more in line with the estimations suggested by [18] than our previous estimations using the data provided by [1]. Further economical and ecological research is necessary to estimate how good these estimations are, but we presented them as an exercise in estimation in costs driven by public perception of the problem, taking in account the data presented by [18], and the different hypothesis of the estimated costs we assumed.

## 5 Discussion

### 5.1 Main Conclusions

Our results for the work seem to indicate that the policy of restricting the controlled space by gaining a better rate of culling seem to be useful when low maximum rates are considered. This result proved to be true by taking the values proposed by [1], and also with our cost estimation functions we proposed using the data provided by [18]. Of course, the considerations for these ecological costs results are, to the present, sketchy at best, but they represent an interesting result to be further explored not just for this particular problem, but to other similar control problems, in which we can stress control policies reducing the space controlled.

During the development of the present work we had the opportunity to discuss with people who dedicate to trapping, and in their opinion, the main issue for an effective trapping strategy is not so much issues of equipment, but the skill of the trapper. Our analysis assumed low maximum trapping rates, given the small population and lack of trapping culture in Tierra del Fuego. The development of effective control policies for the control of the Beaver Infestation must take in account this important factor.

### 5.2 Open problems and Alternative Considerations

- For simplicity sake, and time constraints, the analysis we made was restricted to 1-D, and just taking data from Navarino Island. Extending this analysis to two dimensions seems to be the next step, including in the modeling geographical considerations relevant

to the area, including natural barriers and improving the parameters, adding restrictions to the growth parameters in function of terrain restrictions, or availability resources.

- Extending the space to 2-D also will have an effect in the shape of the controlled space, and how that factor will affect the results, in particular, to take our criteria of restricting the controlled space and allowing greater culling ratios.
- Using the mathematical results obtained to design better strategies, such as, using different controls in different zones, and also, different control strategies at different time frames, which also would involve the problem of selecting optimal switching times of strategies.
- Better estimation of cost functionals. Our mathematical analysis allows for interpolation of any polynomial cost function we may require, so more precise cost functionals must be considered. Our present analysis must be considered an sketch, at best, for a larger discussion in terms of beaver control policies which are necessary both in Chile and Argentina to deal with this infestation.

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## 7 Appendices

### 7.1 Appendix A: Pseudocode for PDE Backward-Forward Algorithm

The following is a description of the Backward-Forward algorithm, as was programmed for this work. For further reference about this algorithm, refer to Chapter four in [5] and [9].

**Data:**  $\epsilon$ : Tolerance

**Result:**  $u_f$ : Optimized control.

initialization;

$u = 0$ ;

$N = 0$ ;

$\lambda = 0$ ;

exit=false;

**while**  $exit = false$  **do**

$N_{new} = SolveTrajectory(u)$ ;

$\lambda_{new} = SolveAdjoint(u, N)$ ;

$u_{new} = Formula(N_{new}, \lambda_{new})$ ;

$err_1 = ||u - u_{new}|| - \epsilon ||u||$ ;

$err_2 = ||\lambda - \lambda_{new}|| - \epsilon ||\lambda||$ ;

$err_3 = ||N - N_{new}|| - \epsilon ||N||$ ;

**if**  $err_1 < 0$  **and**  $err_2 < 0$  **and**  $err_3 < 0$  **then**

$u_f = u$ ;

        exit = true ;

**else**

$u = u_{new}$ ;

$N = N_{new}$ ;

$\lambda = \lambda_{new}$ ;

**end**

**end**

#### **Algorithm 1:** Backward Forward Algorithm

In the preceding algorithm,  $SolveTrajectory(u)$  and  $SolveAdjoint(u, N)$  solve numerically the PDEs using methods such as Runge-Kutta. In our work, this method is applying for solving equations such as 2, 5 with 3 as objective function, for example.