Dynamics of Voter Satisfaction in Multi-Party Systems: A Mathematical and Simulation-Based Approach

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Abstract

We analyzed the effects of ideological changes in individuals and political parties on voter satisfaction. We also examine the impact on satisfaction of changes in parties' positions and how polarization may arise. We created an 2-dimensional ideological trait space, within which we may position individuals and political parties based on their position for each trait. We then created mechanisms for individuals and parties to update their ideological positions. We studied the effects of these behaviors on voter satisfaction. Our model is based on functions that update the positions of individuals and parties. We let the parties that individuals supports and vote for change as their ideological positions change. Finally, a function calculates voter satisfaction over time, based on the ideological distances between individuals and parties' platforms. We looked for and analyzed trends in voter satisfaction and party polarization by running thousands of simulations. We observed that voter satisfaction increases with the number of parties. We also observed that initial positions determine whether the movement of parties polarizing.

Keywords: Voter Satisfaction, Polarization, Ideology, Multi-Party systems

1 Introduction

In the increasingly globalized world of politics, elections and the voting systems that select government leaders and public officials have been at the forefront of recent political discourse. Voting systems—methods by which voters preferences are marked and counted to elect an individual or group of officials—are a crucial concern for many voters in their respective regions' *constituencies* (individuals who vote for a given party or candidate) [9]. The election of these officials often affects the lives and interests of millions in many countries with varying voting systems.

For this study, we defined a *voting system* as a method/algorithm that elects a candidate, party, or group of officials for public office. Many types of voting systems operate in different ways, which directly or indirectly allow constituents to vote based on their ideological preference and values. For example, *ranked-choice voting* is a preference system where constituents rank their preferred candidates through many rounds until an individual/party receives a majority of votes. This system is used in several cities, states, and countries, including Maine, and Alaska, as well as Cambridge, San Francisco, and New York City. *Single transferable voting* (STV) proposes a group of representatives that reflects the ideological diversity of the region. This system enables constituents within a party to select among a variety of candidates from the same or different parties. One of the unique features of STV is that it allows an individual's vote to be transferred to other candidates if their top choice has already secured enough votes to win or if their candidate has been eliminated in an earlier round. It is argued that this system of voting gives an individual voter the most power over their vote. It is used in the United Kingdom, Northern Ireland, the Republic of Ireland, Malta, Scotland, and Australia [18].

A *party* is defined as a political group for which a candidate for parliament can receive votes. In many elections —formal and informal—the parties are a quintessential element of voting systems. In some countries, a two-party system exists, where two dominant political parties elect officials for many legislative positions. Many countries, in contrast, have a multi-party system, where more than two parties elect officials to parliamentary positions. These highly-regarded positions are held by a party or coalition of parties, gaining influence on legislation, social, and judicial measures based on their ideology.

Voting systems are meant to elect officials who propose and enact policies, and aim to represent the people and their ideological values at a given time. However, it is fair to ask whether voting systems actually represent public opinion. In recent years, voters' trust in elected officials has been stagnant or nonexistent. A June 2024 Pew Research Center study found that only 22% of Americans say they trust the federal government to do "what is in the best interest of the people." This reveals the growing discontent and disconnect that many constituents from both the developed and developing world have felt, especially within binary/two-party electoral systems [5].

The phenomenon of shifting constituencies and party ideologies has been documented in previous literature. Downs (1957) proposed his Downsian spatial model of voting, using a 1-dimensional ideological spectrum that plots each constituent at an ideal point in space, assuming that voters choose the party that best aligns with their interests. In response to such a choice, parties move towards a "center" within the ideological space to win the most votes. Downs is characterized by the Median Voter Theorem, which is commonly used in other political literature. The Median Voter Theorem explains certain game-theoretical aspects of party competition in a two-party electoral system. However, it fails to account for other aspects in multi-party elections besides empirical assumptions, including (a) that all parties eventually converge to a certain stable point in a 1-D ideological spectrum, and (b) that constituents vote on ideological proximity [19].

Bernauer & Vatter further emphasize how electoral systems affect satisfaction within a current administration. They studied twenty-four different countries, including the United States, Germany, France, South Africa, and Spain, using survey data from the Comparative Study of Electoral Systems [23]—a dataset containing constituents' survey responses and demographics for election cycles from 1996 to the present day. For their study, they used the democratic satisfaction section, which records individuals' responses using a Likert scale from zero to ten, where zero equals strongly dissatisfied and ten equals strongly satisfied, and a country-level satisfaction given by the governing administrations. They took the individual-level data, known as Level 1, and the country-level data, known as Level 2, to construct their logistic multilevel model to test how these variables affect satisfaction. Their equation goes as follows: $logit(P(Y_{ij} = 1)) = \beta_0 + \beta_1 X_{ij} + \beta_2 Z_j + u_j$ where the logarithm of the probability of being satisfied with a system P for the citizen i in the country j is

modeled by linear regression with intercept β_0, a matrix containing individual-level covariates, X, and a vector of country-level covariates, Z. They found that consensus systems (those focused on collectivity between multiple parties) reduce the overall "winner-loser" gap in voting satisfaction. This reveals that satisfaction is higher in countries with more parties. While Bernauer and Vatter support our initial hypotheses, they used data focused on bigger themes for satisfaction, such as democracy. Additionally, the surveys used in the CSES do not focus on more nuanced topics that constituents consider essential for why they voted in a particular way —such as economics, social issues, foreign policy, etc.— due to the vagueness of the Likert scale. However, that article gives us the context and an initial idea of how to define our model.

[15] classified parties by their platform adaptivity as follows:

(i) **Sticker:** never change position (an 'ideological' party leader).

(ii) **Aggregator:** set party policy on each dimension at the mean preference of all party supporters (a 'democratic' party leader).

(iii) **Hunter:** if the last policy move increased support, make the same move; else, reverse heading and make a unit move in a heading chosen randomly from the arc G908 from the direction now being faced (an autocratic party leader who is a Pavlovian vote-forager).

(iv) **Predator:** identify largest party; if this is you, stand still; else, move towards largest party (an autocratic party leader who seeks votes by attacking larger parties).

[16] argue that "two important human action selection processes are the choice by citizens of parties to support in elections and the choice by party leaders of policy 'packages' (platforms) offered to citizens in order to attract this support." After analyzing these choices and the reasons for doing this, they decided to "treat the number and identity of political parties as an output of, rather than an input to, the process of party competition." They further state that politicians/parties are concerned with maximizing political support while citizens must choose one of the policy packages on offer. The problem for politicians/parties is to offer a policy package that maximizes citizen support, subject to certain constraints. The authors also illustrated the movement of parties' supporter base in a multi-dimensional ideological space and assumed the existence of a threshold for a minimal percentage of votes captured by a party to survive. Their agent-based simulations revealed the impact of party competition in a multi-party system on the number of long-term surviving parties. They found "that, if the survival threshold is 0.20 (20%) of the vote, which tends

to result in 3- or 4-party systems. A 15% threshold tends to result in a 4- or 5-party system, the 10% threshold in 6- or 7- party systems, the 7.5% threshold in 8–10-party systems and the 5% threshold in systems with 12–15 parties. This gives us a sense, for example, that real electoral systems with six or seven parties must have a de facto survival threshold; however, this might be realized in practice, of approximately 10 per cent."

Voters in a dynamic system like an election might not vote based on their proximity to their preferred party or candidate. Additionally, voters may not vote preferentially, choosing a party that is not completely aligned with their ideological values but may likely win an election [27]. The effects of peer pressure from a political opinion leader have been extensively studied, revealing that some voters gain satisfaction by voting according to a political opinion leader, as it is part of their civic duty, further increasing polarization, a division caused due to different political ideologies [10, 11, 22].

In a multi-party parliamentary system where seats are distributed based on vote percentages, constituencies may feel better represented because their party would always represent them in parliament in proportion to the size of the constituencies. This contrasts with a strictly two-party winner-takes-all system, where minority party voices are often excluded or underscored.

Strategy among parties has shifted toward capturing the largest number of votes in their constituency, often at the expense of representing their voter base accurately. Game-theoretic tactics, such as proposing centrist policies, are used to attract a broader spectrum of voters in the ideological space. [19] describes this as an "equilibrium between parties" akin to a tug-of-war game rather than a genuine effort to address constituents' needs. This party competition has led to polarization, which affects overall voter turnout. Related to this issue, [13] emphasized that, based on survey data from 24 different panel studies in the dataset of Comparative Studies in the Electoral System (CSES), one can surmise that voter turnout and voter satisfaction are significantly correlated [3]. With the line differentiating between many parties and constituencies becoming more blurred, voter turnout could decrease and negatively affect satisfaction.

Other studies also reveal that voter satisfaction and turnout are positively correlated. For instance, the 2024 U.S. Presidential Election had lower turnout than the one in 2020, despite

pandemic and recession conditions in the earlier year. Similarly, EU elections saw record-low turnout, despite ideological extremism and a multi-party structure. This suggests that even in multi-party systems, polarization and ideological extremes can alienate voters. This notion is proposed by those who challenge the conventional belief that satisfaction with democracy increases voter turnout [4]. Using the "Making Electoral Democracy Work Project" created by [3], the study uses twenty-four regional election samples from seventeen elections between 2010 and 2015. First, the authors analyzed pre-election satisfaction using a Likert scale that asked participating constituents from both the winning and losing sides to describe how they felt about the election and the incoming administration on a scale of zero to ten. In their survey, they recorded several demographic controls including age, sex, interest in politics, and feeling of closeness to a party. Lastly, to show the timeliness of their study, these variables were tracked throughout the incoming administration's time in power. They recorded these measures over five years and used the regression model $S_{i,t+1} = \alpha + \beta_1 V_{i,t} + \beta_2 S_{i,t} + \epsilon$, where the voter's satisfaction with the ruling government in the next time step (in this case, an election cycle) depends linearly on their satisfaction at the previous cycle and voter turnout at $V_{i,t}$. They documented that voter participation increases a multitude of aspects of voting, like satisfaction. Within their model, the pre-election and initial condition states were analyzed, as well as what happens simultaneously during the administration. However, this paper defines voter satisfaction as a qualitative abstraction from the survey data, referred to as a general feeling of satisfaction; there is no mathematical model that supports it quantitatively. Additionally, there is some subjectivity when it comes to analyzing the variables in the dataset.

Previous research explores voter satisfaction across different electoral systems using empirical metrics [9]. To do this, they used survey data from the CSES dataset, and then used their index to label countries based on the voting system that they are currently using. The authors focused on thirty countries, including Ireland, Japan, and Romania, and analyzed their electoral systems and how they correlate to voter satisfaction. However, little research analyzes voter satisfaction and other trends over discrete time frames.

Mathematical models in politics, such as the one in [7, 8], have laid the groundwork for un-

derstanding ideological alignment between voters' ideology and parties' platforms. Downs' model calculates mean Euclidean distances in a 1-dimensional trait space acting as a proxy for the political spectrum—conservative to liberal—in which a voter moves. Other models incorporate peer pressure and dynamic system modeling to estimate voter turnout and long-term satisfaction.

Our goal: To estimate how the incremental, stepwise adjustment of individual and party ideological positions influences the short- and long-term average satisfaction of each party's constituency as well as that of the whole population. We also aim to analyze the convergence or divergence of party positions given voter preference distributions in multi-party systems. Using our Euclidean-distance model, we document their effects on constituencies and explore how the number of parties impacts ideological alignment.

2 Methodology

To systematically capture the co-evolution of voter preferences and party platforms, we represent both as points in a shared 2-dimensional ideological trait space, $[0, 1]^2$. Each axis corresponds to a central issue or policy for individuals and parties that is represented in a quantitative scale normalized to the interval $[0, 1]$(for example, economic: left–right, social: conservative–liberal, etc.). By assigning every voter, indexed by i, a vector function in the ideological trait space, $\mathbf{x}_i^{(t)}$ representing their evolving ideology, and every party, indexed by j, an evolving platform/ideological-position $\mathbf{p}_j^{(t)}$, we obtain a unified geometric framework in which:

- **Euclidean distances** directly quantify ideological proximity
- **Ideological vector updates** naturally capture gradual individual or party ideological shifts as parties respond to their voter bases and voters respond to peer and party influences.
- **Aggregate measures** of distance (e.g., mean distance between a voter's ideological position and each party's platform) provide interpretable metrics of overall and per-party voter satisfaction.

This *ideological space* formalism allows us to define precise update rules and satisfaction functions, and to analyze their joint dynamics —analytically and through simulation. Specifically, we proceed as follows:

1. **Initialization.** We consider N voters and generate their ideological positions as $(M+1)$–component vectors with M Gaussian components and a uniform floater background, then shift party platforms toward the electorate mean. By doing this, it provided an initial condition for our simulations, meant to represent constituencies and parties given various voting systems (Binary and/or Non-binary systems). To initialize our ideological space, each party is assumed to have a dense core of committed supporters clustered near a central point, plus a sprinkling of undecided "floaters" scattered uniformly across $[0,1]^2$. We therefore draw most voters from Gaussian blobs centered at each party's core, with covariance tuned tighter for extremes and looser for centrists, and draw the remainder uniformly. This is meant to correspond to realistic settings found in real-life elections where a party's voter base do not all have the same ideological allegiance or values to a given party. Finally, each party's starting platform is set partway between its core center and the global mean, so that it initially courts both its base and the general electorate =:

$$(\pi_1,\dots,\pi_M)\sim \text{Dirichlet}(\alpha,\dots,\alpha),\quad \pi_{M+1}=p_{\text{float}},\quad \sum_{j=1}^{M+1}\pi_j=1,$$

$$z_i\sim\text{Categorical}(\pi_1,\dots,\pi_{M+1}),\quad \mathbf{x}_i^{(0)}\mid z_i=j\sim\begin{cases}\mathcal{N}\big(\mu_j,\ \sigma_{\text{base}}^2(1+c_j)I_2\big), & 1\le j\le M,\\ \text{Uniform}([0,1]^2), & j=M+1,\end{cases}$$

with each μ_j arranged on a circle in $[0,1]^2$ and $c_j=c_{\text{edge}}$ or c_{center}. Clip all $\mathbf{x}_i^{(0)}$ to $[0,1]^2$, then set

$$\bar{x}=\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i^{(0)},\qquad \mathbf{p}_j^{(t+1)}=(1-\eta-\kappa-\sigma-\rho)\,\mathbf{p}_j^{(t)}+\eta\Big(\bar{\mathbf{x}}_{S_j^{(t)}}-\mathbf{p}_j^{(t)}\Big)+\kappa\sum_{k\neq j}W_{jk}\Big(\mathbf{p}_k^{(t)}-\mathbf{p}_j^{(t)}\Big)+\sigma\,\xi_j^{(t)}+\rho\mathbf{r}_j,$$

$j=1,\dots,M.$

2. **Distance Metrics.** To determine how close two ideological positions **a** and **b** are at any given

time t, we use the Euclidean distance between them in the ideological space. $\mathscr{I} = [0,1] \times [0,1]$:

$$d_{ij}^{(t)} = \|\mathbf{a} - \mathbf{b}\|.$$

Here $\mathbf{a}$ and $\mathbf{b}$ may refer to two voters' positions, two parties' platforms, or one voter's position and one party's platform.

3. **Party-Position Update.**

$$\mathbf{p}_j^{(t+1)} = (1 - \eta - \kappa - \sigma - \rho)\,\mathbf{p}_j^{(t)} + \eta\left(\bar{\mathbf{x}}_{S_j^{(t)}} - \mathbf{p}_j^{(t)}\right) + \kappa \sum_{k \neq j} W_{jk}\left(\mathbf{p}_k^{(t)} - \mathbf{p}_j^{(t)}\right) + \sigma\,\xi_j^{(t)} + \rho\,\mathbf{r}_j, \quad (1)$$

where

$$\bar{\mathbf{x}}_{S_j^{(t)}} = \frac{1}{|S_j^{(t)}|} \sum_{i \in S_j^{(t)}} \mathbf{x}_i^{(t)},$$

and

$$\mathbf{r}_j = \begin{cases} \dfrac{\mathbf{p}_{j_1}^{(0)} - \mathbf{p}_{j_2}^{(0)}}{\|\mathbf{p}_{j_1}^{(0)} - \mathbf{p}_{j_2}^{(0)}\|}, & j = j_1, \\ \dfrac{\mathbf{p}_{j_2}^{(0)} - \mathbf{p}_{j_1}^{(0)}}{\|\mathbf{p}_{j_2}^{(0)} - \mathbf{p}_{j_1}^{(0)}\|}, & j = j_2, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

Parties move as a convex combination of five philosophical forces:

- *Inertia* $(1 - \eta - \kappa - \sigma - \rho)\,\mathbf{p}_j^{(t)}$: retains institutional memory and ideological consistency, reflecting the party's commitment to its past platform.
- *Support-Pull* $\eta\big(\bar{\mathbf{x}}_{S_j^{(t)}} - \mathbf{p}_j^{(t)}\big)$: embodies democratic responsiveness—parties shift toward the average views of their active supporters, honoring grassroots feedback.
- *Competition* $\kappa \sum_{k \neq j} W_{jk}(\mathbf{p}_k^{(t)} - \mathbf{p}_j^{(t)})$: encodes strategic differentiation or clustering, modeling the dialectic between ideological mimicry and distinction in a multiparty arena.
- *Random Drift* $\sigma\,\xi_j^{(t)}$: captures unforeseen shocks—scandals, crises, charismatic interventions—that can jolt platforms unpredictably.

- *Repulsion* $\rho\,\mathbf{r}_j$: enforces durable polarization by pushing the two initially farthest parties further apart, symbolizing entrenched ideological antagonism.

Together, these terms philosophically represent how parties balance fidelity to their history, commitment to their base, strategic positioning relative to rivals, adaptability under uncertainty, and the centrifugal forces of polarization to maximize electoral support in the ideological marketplace.

4. **Voter-Position Update.**

$$\mathbf{x}_i^{(t+1)} = (1-\alpha-\beta)\,\mathbf{x}_i^{(t)} + \alpha\,\bar{\mathbf{x}}_{\mathcal{N}_i^{(t)}} + \beta\,\mathbf{p}_{k_i}^{(t)}, \tag{2}$$

where

$$\mathcal{N}_i^{(t)} = \{\,\ell : \|\mathbf{x}_\ell^{(t)} - \mathbf{x}_i^{(t)}\| \le \epsilon\}, \qquad k_i = \arg\min_k d_{ik}^{(t)}.$$

Voters reposition themselves as a convex combination of three philosophical drives:

- *Inertia* $(1-\alpha-\beta)\,\mathbf{x}_i^{(t)}$: preserves cognitive consistency and personal conviction, resisting change absent compelling influence.
- *Peer Influence* $\alpha\,\bar{\mathbf{x}}_{\mathcal{N}_i^{(t)}}$: embodies social conformity and the desire to align with one's immediate community, reflecting trust in local norms and discussion networks.
- *Party Loyalty* $\beta\,\mathbf{p}_{k_i}^{(t)}$: captures strategic alignment with the closest party's platform, modeling the calculus between individual belief and perceived political efficacy.

Together, these terms philosophically represent how voters balance steadfast personal beliefs, local social pressures, and partisan identification when updating their ideological stance.

5. **Party Switching.** Assign each voter i to the nearest party $\pi(i;t) = \arg\min_{1\le j\le M} d_{ij}^{(t)}$, defining $S_j^{(t)} = \{\,i : \pi(i;t) = j\}$.

Given an updated party affliction, at each timestep t, each voter, $x_i^{(t)}$ now checks which party, $P_j^{(0)}$ at given t is closest in the ideological space. The voter will then reassign their ideological

position to the now closest party if that party is different from their prior affiliation.

$$j_i^{(t)} = \arg\min_{1 \le j \le M} \left\| \mathbf{x}_i^{(t)} - \mathbf{p}_j^{(t)} \right\|_2 \tag{3}$$

Party switching is when a voter decides to switch their allegiance and move to a different party. This could be because the party best aligns with their ideological values at a given timestep t. We see this a lot more in parliamentary systems where forming coalitions and switching party allegiance are common. Realistically, voters' ideological values evolve over time. Party switching is more than a strategic or ideological realignment; it recognizes that voters' opinions are not fixed and can slowly or rapidly change per election cycle, representing a voter's vision on their immediate society and environment. However, switching parties could be used in a game theory sense, whether voters switch their party allegiance to vote for a party that might not have the best ideological fit, but is somewhat close to them and likely to win. Our model simulation recomputes voters and switches their party allegiance based on their distance to all other party platforms, whose platform is closest to their own ideological position. This is meant to illustrate how voters eventually evolve their ideological values and go to parties that represent them the best. This is especially prevalent in multi-party systems and parliaments.

6. **Satisfaction**

$$\mathrm{VS}_i^{(t)} = \sum_{j=1}^{M} w_{ij}^{(t)} \frac{10}{\sqrt{2}} \left(\sqrt{2} - d_{ij}^{(t)}\right), \qquad \in [0, 10], \tag{4}$$

$$\mathrm{VS}_j^{(t)} = \frac{1}{|V_j^{(t)}|} \sum_{i \in V_j^{(t)}} \mathrm{VS}_i^{(t)}, \qquad V_j^{(t)} = \{\, i : \arg\min_k d_{ik}^{(t)} = j \}, \tag{5}$$

$$\overline{\mathrm{VS}}^{(t)} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{VS}_i^{(t)}. \tag{6}$$

Voters' satisfaction is how satisfied/happy a voter is with the current voting system. Over time, especially during a specific current reigning administration, voters' opinions on how the

electorate or policy within a certain electoral system are. Literature suggests that voters' satisfaction with an administration often dwindles over time. Like other aspects related to voting, opinions, feelings, and thoughts change. With this in mind, we built our model simulation to encapsulate this phenomenon. Rather than treating satisfaction as a static term, our approach includes time-based variation to analyze satisfaction between policy and ideology. By modeling these changes, we aim to gain a deeper understanding of how people view a voting system/regime is built or eroded over time, and how voting systems might better respond to the lived experiences of the constituency and population parties and electorates are meant to serve.

For the individual voter satisfaction. Each voter within their constituency looks at where all the parties are in the ideological space. If a given party $P_j^{(0)}$ is far from a given voter $x_i^{(t)}$, the voter is less satisfied. If a party is very close to a voter, then the voter would be nearly perfectly happy. When looking at party satisfaction, we took the average of how satisfied the party's voter base is. Finally, we took the overall satisfaction of all the voters given a particular voting system. This is meant to track voter representation, how approved are a party's policies, etc.

7. **Monte Carlo Simulations.** Compute long-term satisfaction and polarization for many random seeds and average the results from these runs to study how the number of parties, M, and other parameters (e.g., $\eta, \kappa, \sigma, \alpha$) impact those.

2.1 Realistic Initialization

In order to capture the dual nature of American-style electorates—large, tightly-bound cores of committed supporters plus a broad pool of loosely-affiliated "floaters"—we initialize each voter's ideological position $\mathbf{x}_i^{(0)} \in [0,1]^2$ from an $(M+1)$–component mixture model, rather than the simplistic uniform distribution. Concretely:

$$\mathbf{x}_i^{(0)} \sim \begin{cases} \mathcal{N}\big(\mu_j, \Sigma_j\big), & j = 1, \ldots, M, \\ \text{Uniform}\big([0,1]^2\big), & j = M+1, \end{cases} \qquad \Pr[z_i = j] = \pi_j,$$

where z_i indicates which component (party core or floater) generated voter i. We draw the mixture weights

$$(\pi_1, \ldots, \pi_M) \sim \text{Dirichlet}(\alpha, \ldots, \alpha), \quad \pi_{M+1} = 1 - \sum_{j=1}^{M} \pi_j,$$

so that α controls how evenly the electorate is split among the M party cores (smaller α produces more uneven, highly-polarized cores, while larger α yields more balanced support). The remaining probability π_{M+1} is assigned to the uniform "floater" component, representing voters whose ideological leanings are weak or undecided.

Each Gaussian core $\mathcal{N}(\mu_j, \Sigma_j)$ models the concentrated cluster of strong partisans around a central "platform" location μ_j. The covariance

$$\Sigma_j = \sigma_{\text{base}}^2 \, (1 + c_j) \, I_2 \quad \text{with} \quad c_j = \begin{cases} c_{\text{edge}}, & j \in \{\text{extreme-wing parties}\}, \\ c_{\text{center}}, & j \in \{\text{centrist parties}\}, \end{cases}$$

allows "edge" parties (those on the ideological fringes) to have tighter, more homogeneous cores ($c_{\text{edge}} \ll 1$) while centrist parties draw from a broader, more diffuse base ($c_{\text{center}} > 1$). This reflects empirical observations that polarization often produces smaller but more cohesive extremist blocs, whereas moderates span a wider range of views.

After sampling all N voters, we clip any points lying outside $[0,1]^2$ back to the boundary, then calculate the overall mean position of the electorate. Each party updates its ideological position based on the update mechanism from equation (4).

This initialization serves two purposes:

1. **Empirical realism:** It mirrors the way real political bases form around clear ideological centers with highly motivated supporters, while a sizable fraction of voters remain unaligned and swing between parties.

2. **Dynamic flexibility:** By starting platforms between core and mean positions, parties are endowed with room to evolve, either consolidating around their base or seeking broader appeal as the simulation unfolds.

With this setup, subsequent party position updates (Equation 1) and voter position updates operate on a realistically structured electorate, ensuring that our simulation captures both the hard-core dynamics and the fluidity of the political center.

2.2 Party Position Update Rule

Once voter–party affiliations are determined, each party j revises its platform by balancing five conceptual forces: institutional inertia, base responsiveness, strategic positioning relative to rivals, stochastic shocks, and a built-in polarization push between the two initially most distant parties. Formally,

$$\mathbf{p}_j^{(t+1)} = \underbrace{(1-\eta-\kappa-\sigma-\rho)\,\mathbf{p}_j^{(t)}}_{\substack{\text{Inertia}\\ \text{(historical consistency)}}} + \underbrace{\eta\left(\bar{\mathbf{x}}_{S_j^{(t)}} - \mathbf{p}_j^{(t)}\right)}_{\substack{\text{Support-Pull}\\ \text{(grassroots feedback)}}} + \underbrace{\kappa \sum_{k\neq j} W_{jk}\left(\mathbf{p}_k^{(t)} - \mathbf{p}_j^{(t)}\right)}_{\substack{\text{Competition}\\ \text{(differentiation vs. clustering)}}}$$
$$+ \underbrace{\sigma\,\xi_j^{(t)}}_{\substack{\text{Random Drift}\\ \text{(unpredictable shocks)}}} + \underbrace{\rho\,\mathbf{r}_j^{(t)}}_{\substack{\text{Repulsion}\\ \text{(entrenched polarization)}}} \tag{7}$$

Interpretation:

- *Inertia* $(1-\eta-\kappa-\sigma-\rho)\,\mathbf{p}_j^{(t)}$: Models the party's institutional memory and ideological consistency, reflecting reluctance to abandon core convictions without sufficient impetus.
- *Support-Pull* $\eta\,(\bar{\mathbf{x}}_{S_j^{(t)}} - \mathbf{p}_j^{(t)})$: Moves the platform toward its active supporters' centroid, embodying democratic responsiveness to grassroots feedback.
- *Competition* $\kappa \sum_{k\neq j} W_{jk}(\mathbf{p}_k^{(t)} - \mathbf{p}_j^{(t)})$: Encodes strategic positioning—differentiation when moving away from competitors or clustering toward successful peers—reflecting real-world coalition and identity dynamics.

- *Random Drift* $\sigma\,\xi_j^{(t)}$: Captures exogenous shocks (scandals, crises, leadership changes) that can unpredictably jolt a party's platform.
- *Repulsion* $\rho\,\mathbf{r}_j^{(t)}$: Applies only to the two parties (j_1, j_2) with maximum initial separation, pushing them farther apart to model entrenched polarization:

$$\mathbf{r}_j^{(t)} = \begin{cases} \dfrac{\mathbf{p}_{j_1}^{(0)} - \mathbf{p}_{j_2}^{(0)}}{\|\mathbf{p}_{j_1}^{(0)} - \mathbf{p}_{j_2}^{(0)}\|}, & j = j_1, \\ \dfrac{\mathbf{p}_{j_2}^{(0)} - \mathbf{p}_{j_1}^{(0)}}{\|\mathbf{p}_{j_2}^{(0)} - \mathbf{p}_{j_1}^{(0)}\|}, & j = j_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Philosophical meaning: Equation (7) portrays each party's evolution as a nuanced balancing act:

- *Fidelity vs. Flexibility:* Inertia and support-pull negotiate between steadfast principle and adaptation to voter demands.
- *Distinctiveness vs. Convergence:* Competition and repulsion govern whether parties differentiate themselves or seek coalition common ground.
- *Predictability vs. Chaos:* Random drift acknowledges that political change can be sudden and unpredictable.

Significance: This update rule is the engine driving our model's emergent dynamics. By iterating Eq. (7) alongside the voter update (Eq. (8)), we generate:

- *Feedback loops:* Voter realignments reshape support bases $S_j^{(t)}$, which in turn drive subsequent platform shifts.
- *Satisfaction trajectories:* Moving platforms alter voter–party distances $d_{ij}^{(t)}$, affecting individual and aggregate satisfaction metrics over time.
- *Parameter insights:* Systematic variation of $\eta, \kappa, \sigma, \rho$ reveals how responsiveness, competition, randomness, and enforced polarization each influence long-run satisfaction and party viability.

- *Policy experimentation:* The transparent decomposition of forces offers a conceptual guide for assessing how institutional tweaks (e.g. raising thresholds to modulate ρ) or media volatility (σ) might mitigate polarization and enhance voter satisfaction.

2.3 Individual Ideological Position Dynamics

Voter opinions evolve under four conceptual forces—personal inertia, peer influence, party loyalty, and stochastic perturbations—in a two-dimensional issue space $\mathbb{R}^2$. At each discrete time step t, voter i updates their position $\mathbf{x}_i^{(t)}$ by blending:

$$\mathbf{x}_i^{(t+1)} = \underbrace{(1-\alpha-\beta_i)\,\mathbf{x}_i^{(t)}}_{\substack{\text{Self-Consistency}\\ \text{(ideological inertia)}}} + \underbrace{\frac{\alpha}{|\mathcal{N}_i^{(t)}|}\sum_{j\in\mathcal{N}_i^{(t)}} \mathbf{x}_j^{(t)}}_{\substack{\text{Peer Influence}\\ \text{(social conformity)}}} + \underbrace{\beta_i\,\mathbf{p}_{k_i}^{(t)}}_{\substack{\text{Party Loyalty}\\ \text{(strategic alignment)}}} + \underbrace{\sigma\,\boldsymbol{\xi}_i^{(t)}}_{\substack{\text{Random Drift}\\ \text{(exogenous shocks)}}}, \tag{8}$$

where:

$$\mathcal{N}_i^{(t)} = \{\, j : \|\mathbf{x}_j^{(t)} - \mathbf{x}_i^{(t)}\| \le \varepsilon\} \quad \text{and} \quad k_i = \arg\min_k d_{ik}^{(t)}.$$

Interpretation

- *Self-Consistency* $(1-\alpha-\beta_i)\,\mathbf{x}_i^{(t)}$: Maintains continuity of personal conviction and cognitive consistency, modeling resistance to abrupt ideological shifts.
- *Peer Influence* $\frac{\alpha}{|\mathcal{N}_i^{(t)}|}\sum \mathbf{x}_j^{(t)}$: Captures social conformity within a voter's local neighborhood, reflecting the tendency to align with trusted peers.
- *Party Loyalty* $\beta_i\,\mathbf{p}_{k_i}^{(t)}$: Pulls the voter toward the platform of the closest party, embodying strategic support for a party perceived to best represent one's interests.
- *Random Drift* $\sigma\,\boldsymbol{\xi}_i^{(t)}$: Introduces exogenous shocks—news events, personal experiences, or charismatic appeals—acknowledging the unpredictability of opinion shifts.

Properties and edge cases

- **Convexity:** When $\alpha+\beta_i \le 1$, the deterministic update is a convex combination, ensuring $\mathbf{x}_i^{(t+1)}$ stays within the convex hull of prior beliefs, peer means, and party platforms.

- **Empty neighborhood ($\mathcal{N}_i^{(t)} = \emptyset$):** Peer influence vanishes, leaving only inertia, party loyalty, and noise.
- **Zero party weight ($\beta_i = 0$):** No partisan pull; dynamics reduce to peer averaging plus inertia.
- **Frozen opinion ($\alpha + \beta_i = 0$):** Only stochastic drift applies, locking in deterministic beliefs.

Significance: This individual update rule is foundational to the co-evolutionary project: it determines how voter positions respond to changing party platforms and social context at each time step. By coupling Eq. (8) with the party-position updates (Sec. 2.2), the model captures feedback loops in which:

- Voter shifts alter the support bases $S_j^{(t)}$, influencing subsequent party movements.
- Party adjustments then reshape $\mathbf{p}_j^{(t+1)}$, which feed back into voter realignment in the next iteration.

Together, these dynamics generate emergent phenomena—such as consensus formation, polarization, and satisfaction trajectories—whose behavior over thousands of simulated runs reveals the critical interplay between micro-level opinion changes and macro-level political outcomes. This interplay underpins our analysis of how model parameters (peer influence, party loyalty, noise) drive the evolution of overall and per-party voter satisfaction.

2.4 Voter Satisfaction

Unlike other models found in our literature, voter satisfaction in our model is not simply a fixed rating, but it is a quantitative formula bounded by measures that quantify voters' feelings of allegiance. To do this, we quantified and defined voters' satisfaction by taking the weighted average of voters' proximity to all party platforms, using an exponential distance-based weight. This proximity is computed in a Euclidean policy space, where both voters and parties are represented as vectors of preferences or ideological positions. The closer a voter is to a party platform, the more ideologically aligned they are presumed to be with that party, and the greater their potential satisfaction. To translate these ideological distances into a coherent satisfaction score, we use exponential distance-based weighting, a method that privileges closer parties while still accounting

for the entire spectrum of choices. With this, our satisfaction scores go as follows $S_i \in [0, 10]$ for voter i is defined as

$$S_i \;=\; \sum_{j=1}^{M} \omega_{ij} \, \frac{10}{\sqrt{2}} \big(\sqrt{2} - d_{ij}\big),$$

where:

- $d_{ij} = \|\mathbf{x}_i - \mathbf{p}_j\|$ is the Euclidean distance between voter i and party j.
- $\omega_{ij} \in [0, 1]$ is a softmax weight reflecting voter i's perceived favorability of party j.

The weights ω_{ij} are computed by

$$\omega_{ij} = \frac{e^{(-\lambda \, d_{ij})}}{\sum_{k=1}^{M} e^{(-\lambda \, d_{ik})}},$$

where $\lambda \geq 0$ is a sensitivity parameter:

- As $\lambda \to 0$, $\omega_{ij} \approx 1/M$, and S_i reduces to the average proximity to all parties.
- As $\lambda \to \infty$, the weight concentrates on the closest party, yielding

$$S_i \;\longrightarrow\; \frac{10}{\sqrt{2}} \big(\sqrt{2} - \min_j d_{ij}\big).$$

As λ increases, the weighting of a voter's party preference becomes increasingly concentrated on the closest party platform within a given ideological space. This allows the model to interpolate between:

- **Low discrimination** ($\lambda \to 0$): The voter regards all parties uniformly, regardless of their ideological position and distance. This is meant to reflect the case where voters are tolerant of various aspects, including a lack of information, misinformation, and indifference to voting.
- **High discrimination** ($\lambda \to \infty$): The voter becomes extremely sensitive to ideological position and distance, effectively tying their satisfaction to the nearest party. These individuals can be considered loyalists to that party.

Scaling within our model can capture a range of different voter behavior–rational or non-rational–while documenting its subsequent effects on satisfaction. Overall, our satisfaction model is both behaviorally realistic and computationally tractable, not merely noting where individuals are, but also with what intensity they prefer the alternatives before them.

3 Mathematical Results

Throughout our simulations, both parties and individuals converge toward the geometric center of the given ideological space when using the original party-update rule. This occurs because each party's new position is a convex combination of its previous position and the mean position of its supporters. The rate of responsiveness is controlled by the party-update parameter η: the closer η is to 1, the more responsive parties are to their base and the faster average voter satisfaction peaks.

Individual opinions evolve based on two processes. First, social influence within a fixed ideological radius, ε, weighted by α, causing voters to adopt the average opinion of nearby peers. Second, party influence, governed by β, attracts individuals toward their party's platform. When multiple parties compete, each shifts its position to capture floater and "fringe" voters; the competition parameter, κ, determines how aggressively parties pursue support outside their core voter base. ρ assigns the degree of repulsion between two given parties. Finally, σ introduces random variation into both individual and party updates, simulating how parties and individuals move sporadically given a certain t.

Under our new update equations, the relative magnitudes of α, η, κ, β, and ρ shape the trajectory of the system:

- **High** η (near 1) makes parties very responsive to their base, accelerating convergence and boosting average satisfaction in fewer steps.
- **High** κ drives parties to chase outside/fringe voters more aggressively;. However, this can expand their support; it often reduces satisfaction, both within each party and across the electorate, depending on initial conditions.

- **High** β aligns individuals quickly with their party's position, shrinking the inherent gap between voters and platforms while raising overall satisfaction.
- **High** α accelerates peer influence, which can cause the electorate to settle around a single consensus point (the mean of all individual positions), depending on the initialization of ideological space.
- **High** ρ assigns the level of repulsion, or polarization, that is present between individuals and two or more parties' voter base within an ideological space.
- **High** σ assigns the level of randomness to how parties update their ideological position, this helps to account for the irrationality of how parties change positions update their positions.

By tuning these parameters, we can trade off speed of convergence, breadth of party appeal, and levels of voter satisfaction in rich and nuanced ways to simulate realistic voting factors.

Model Equations

1. **Original Party Update (Eq. 9):**

$$\mathbf{p}_j^{(t+1)} = (1-\eta)\,\mathbf{p}_j^{(t)} + \eta\,\frac{1}{|S_j^{(t)}|}\sum_{i\in S_j^{(t)}}\mathbf{x}_i. \tag{9}$$

2. **Extended Party Update (Eq. 10):**

$$\mathbf{p}_j^{(t+1)} = (1-\eta-\kappa-\sigma)\,\mathbf{p}_j^{(t)} + \eta\big(\bar{\mathbf{x}}_{S_j}^{(t)} - \mathbf{p}_j^{(t)}\big) + \kappa\sum_{k\neq j} W_{jk}\big(\mathbf{p}_k^{(t)} - \mathbf{p}_j^{(t)}\big) + \sigma\,\xi_j^{(t)} + \rho\cdot\mathbf{r}_j \tag{10}$$

3. **Current Individual Update (Eq. 3):**

$$\mathbf{x}_i^{(t)} = (1-\alpha-\beta)\,\mathbf{x}_i^{(t-1)} + \frac{\alpha}{|S_i^{(t)}|}\sum_{\substack{j\in S_i^{(t)}\\ \|\mathbf{x}_j^{(t-1)}-\mathbf{x}_i^{(t-1)}\|<\varepsilon}}\mathbf{x}_j^{(t-1)} + \beta\,\mathbf{p}_{\pi(i,t-1)}^{(t-1)}.$$

(11)

Proposition 3.1. Proof of Convergence to the Geometric center for the old update equation

We now prove that, when voters' positions remain fixed and the original party-update rule is applied, each party's position converges to the centroid of its supporting electorate. First, we derive an explicit closed-form expression for $\mathbf{p}_j^{(t)}$ under the recurrence

$$\mathbf{p}_j^{(t+1)} = (1-\eta)\,\mathbf{p}_j^{(t)} + \eta\,\bar{\mathbf{x}},$$

where $\bar{\mathbf{x}}$ is the mean of the voters in party j. We then show that

$$\lim_{t\to\infty} \left\|\mathbf{p}_j^{(t)} - \bar{\mathbf{x}}\right\| = 0,$$

establishing convergence to the geometric center as t approaches infinity.

Party Position Convergence Model

Let $\mathbf{p}_j^{(t)} \in \mathbb{R}^2$, $\eta \in [0,1]$, and define the group average:

$$\bar{\mathbf{x}} = \frac{1}{|S|}\sum_{i\in S} x_i$$

Assume the update rule:

$$\mathbf{p}_j^{(t+1)} = (1-\eta)\mathbf{p}_j^{(t)} + \eta\bar{\mathbf{x}}$$

We aim to show:

$$\left\|\mathbf{p}_j^{(t)} - \bar{\mathbf{x}}\right\| \to 0 \quad \text{as } t \to \infty$$

Step 1: Derive Closed Form (Geometric Series)

We expand the recurrence for a few steps:

$$
\begin{aligned}
\mathbf{p}_j^{(1)} &= (1-\eta)\mathbf{p}_j^{(0)} + \eta\bar{\mathbf{x}} \\
\mathbf{p}_j^{(2)} &= (1-\eta)\mathbf{p}_j^{(1)} + \eta\bar{\mathbf{x}} \\
&= (1-\eta)^2\mathbf{p}_j^{(0)} + \eta(1-\eta)\bar{\mathbf{x}} + \eta\bar{\mathbf{x}} \\
&= (1-\eta)^2\mathbf{p}_j^{(0)} + \eta(1+(1-\eta))\bar{\mathbf{x}} \\
&= (1-\eta)^2\mathbf{p}_j^{(0)} + \eta[1+(1-\eta)]\bar{\mathbf{x}}
\end{aligned}
$$

Geometric Series Derivation Continued

For $t = 3$, we obtain:

$$p_j^{(3)} = (1-\eta)^3 p_j^{(0)} + \eta[(1-\eta)^2 + (1-\eta) + 1]\bar{x}$$

This suggests the pattern:

$$p_j^{(t)} = (1-\eta)^t p_j^{(0)} + \eta \sum_{k=0}^{t-1} (1-\eta)^k \bar{x}$$

Factor out $\bar{x}$:

$$p_j^{(t)} = (1-\eta)^t p_j^{(0)} + \left(\eta \sum_{k=0}^{t-1} (1-\eta)^k \right) \bar{x}$$

The summation is a geometric series with ratio $r = 1 - \eta$, so:

$$\sum_{k=0}^{t-1} (1-\eta)^k = \frac{1-(1-\eta)^t}{\eta}$$

Thus:

$$p_j^{(t)} = (1-\eta)^t p_j^{(0)} + \left[1-(1-\eta)^t\right] \bar{x}$$

Step 2: Prove the Closed Form by Induction

We now prove that for all $t \in \mathbb{N}$:

$$p_j^{(t)} = (1-\eta)^t p_j^{(0)} + \left[1 - (1-\eta)^t\right] \bar{x}$$

Base case: $t = 1$

$$p_j^{(1)} = (1-\eta)p_j^{(0)} + \eta\bar{x} = (1-\eta)^1 p_j^{(0)} + \left[1 - (1-\eta)^1\right] \bar{x}$$

Inductive hypothesis: Assume the formula holds at time t:

$$p_j^{(t)} = (1-\eta)^t p_j^{(0)} + \left[1 - (1-\eta)^t\right] \bar{x}$$

Inductive step: Show that it holds for $t+1$. Start with recurrence:

$$p_j^{(t+1)} = (1-\eta)p_j^{(t)} + \eta\bar{x}$$

Substitute the inductive hypothesis:

$$p_j^{(t+1)} = (1-\eta)\left[(1-\eta)^t p_j^{(0)} + \left(1 - (1-\eta)^t\right)\bar{x}\right] + \eta\bar{x}$$

Distribute:

$$= (1-\eta)^{t+1} p_j^{(0)} + \left[(1-\eta)\left(1 - (1-\eta)^t\right) + \eta\right] \bar{x}$$

Step 3: Norm Convergence

Subtract:

$$p_j^{(t)} - \bar{x} = (1-\eta)^t (p_j^{(0)} - \bar{x})$$

Taking norms:

$$\|p_j^{(t)} - \bar{x}\| = |1-\eta|^t \cdot \|p_j^{(0)} - \bar{x}\|$$

If $\eta > 0$, then $|1 - \eta| < 1$, so:

$$\text{As } t \to \infty \quad \|p_j^{(t)} - \bar{x}\| \to 0.$$

Edge case: If $\eta = 0$, then no update occurs:

$$p_j^{(t)} = p_j^{(0)} \Rightarrow \|p_j^{(t)} - \bar{x}\| = \text{constant}.$$

Final Remark If $\eta = 0$, then the update rule becomes:

$$\mathbf{p}_j^{(t)} = \mathbf{p}_j^{(0)} \quad \text{for all } t,$$

so that the position of the party remains constant.

Therefore, in the case $\eta = 0$, the only way the position of the party satisfies

$$\mathbf{p}_j^{(t)} \to \bar{\mathbf{x}} \quad \text{as } t \to \infty$$

is if $\mathbf{p}_j^{(0)} = \bar{\mathbf{x}}$ from the beginning.

Proposition 3.2. Proof that $\lim_{\lambda \to \infty} \omega_{ij}$ Converges to the Argmin Indicator

We aim to rigorously prove that the softmax weights:

$$\omega_{ij} = \frac{e^{-\lambda d_{ij}}}{\sum_{k=1}^{M} e^{-\lambda d_{ik}}}$$

converge to an indicator function for the minimum distance as $\lambda \to \infty$, i.e.,

$$\lim_{\lambda \to \infty} \omega_{ij} = \begin{cases} 1 & \text{if } j = \arg\min_k d_{ik} \\ 0 & \text{otherwise} \end{cases}$$

Setup

Fix a voter i and let $\{d_{ij}\}_{j=1}^{M} \subset \mathbb{R}_{\geq 0}$ denote the distances from the voter to each of the M parties.

Define:

$$j^* = \arg\min_j d_{ij} \quad \text{and} \quad \delta_j = d_{ij} - d_{ij^*}$$

so that:

$$\delta_{j^*} = 0, \quad \delta_j > 0 \text{ for all } j \neq j^*$$

Rewrite the Softmax Expression

We substitute the shifted distances into ω_{ij}:

$$\omega_{ij} = \frac{e^{-\lambda(d_{ij^*}+\delta_j)}}{\sum_{k=1}^{M} e^{-\lambda(d_{ij^*}+\delta_k)}} = \frac{e^{-\lambda\delta_j}}{\sum_{k=1}^{M} e^{-\lambda\delta_k}}$$

Limit of the Denominator

We decompose the denominator as follows:

$$\sum_{k=1}^{M} e^{-\lambda\delta_k} = e^{-\lambda\delta_{j^*}} + \sum_{\substack{k=1 \\ k\neq j^*}}^{M} e^{-\lambda\delta_k} = 1 + \sum_{k\neq j^*} e^{-\lambda\delta_k}$$

Each $\delta_k > 0$ for $k \neq j^*$, so:

$$\lim_{\lambda\to\infty} e^{-\lambda\delta_k} = 0$$

Since there are finitely many terms, their sum vanishes:

$$\lim_{\lambda\to\infty} \sum_{k=1}^{M} e^{-\lambda\delta_k} = 1$$

Limit of ω_{ij}

Now consider two cases:

Case 1: $j = j^*$

$$\omega_{ij^*} = \frac{e^{-\lambda \cdot 0}}{\sum_k e^{-\lambda \delta_k}} = \frac{1}{1 + \sum_{k \neq j^*} e^{-\lambda \delta_k}} \to \frac{1}{1} = 1$$

Case 2: $j \neq j^*$

$$\omega_{ij} = \frac{e^{-\lambda \delta_j}}{\sum_k e^{-\lambda \delta_k}} = \frac{e^{-\lambda \delta_j}}{1 + \sum_{k \neq j^*} e^{-\lambda \delta_k}} \to \frac{0}{1} = 0$$

Conclusion

$$\lim_{\lambda \to \infty} \omega_{ij} = \begin{cases} 1 & \text{if } j = j^* \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}_{j = \arg\min_k d_{ik}}$$

This shows that the softmax weight function converges point-wise to an indicator function on the index achieving the minimal distance.

This formulation ensures that $S_i \in [0, 10]$, with 10 representing perfect alignment with the closest party and 0 representing the maximal ideological distance (that is, $d_{ij} = \sqrt{2}$).

Proposition 3.3 (Satisfaction Bound under Distance-Based Weights)**.** Let $\mathbf{x}_i \in [0,1]^2$ be the position of voter i, and let $\mathbf{p}_j(t) \in [0,1]^2$ denote the position of party j at time t. Define the distance between voter i and party j as

$$d_{ij}(t) = \|\mathbf{x}_i - \mathbf{p}_j(t)\|,$$

and the weight assigned to party j by voter i as

$$\omega_{ij}(t) = \frac{e^{-\lambda d_{ij}(t)}}{\sum_{k=1}^{M} e^{-\lambda d_{ik}(t)}}, \quad \text{for some } \lambda > 0.$$

Then the voter's satisfaction at time t is given by

$$S_i(t) = \sum_{j=1}^{M} \omega_{ij}(t) \cdot \left(\frac{10}{\sqrt{2}} (\sqrt{2} - d_{ij}(t)) \right).$$

This function satisfies the bounds

$$0 \leq S_i(t) \leq 10,$$

with the following properties:

- $S_i(t) = 10$ if and only if $d_{ij}(t) = 0$ for some party j, and $\omega_{ij}(t) \to 1$.
- $S_i(t) \to 0$ only in the limit where $d_{ij}(t) \to \sqrt{2}$ for all j.

4 Numerical Simulation

We conducted thousands of simulations using programs of our model functions. The resulting simulations, figures, graphs, and trends allow us to understand and analyze our model.

4.1 Party Update

We have simulated and analyzed the behaviors created by each parameter and mechanism for parties' position updates. The following sets of figures demonstrate the behavior of each parameter, with other parameters held constant.

The first parameter that causes party positions to update is η. This parameter moves parties towards the geometric centroid of their voter base. The following figures demostrates this behavior, with all other parameters disabled.

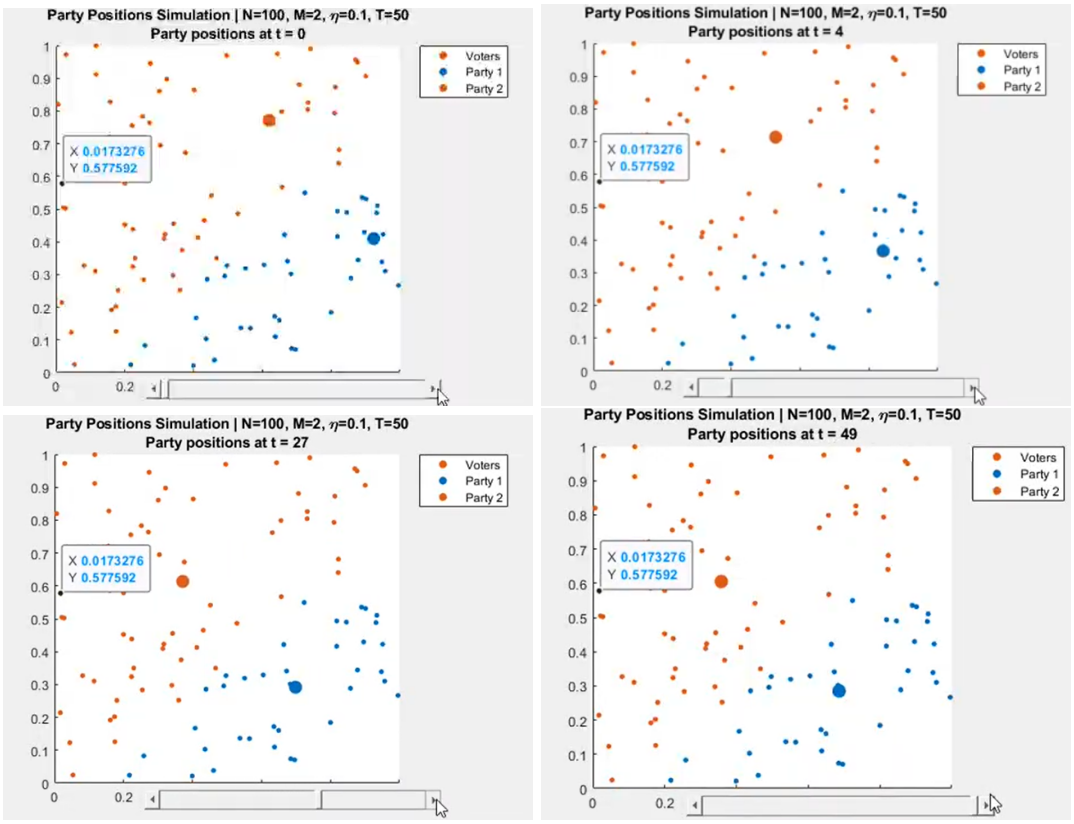
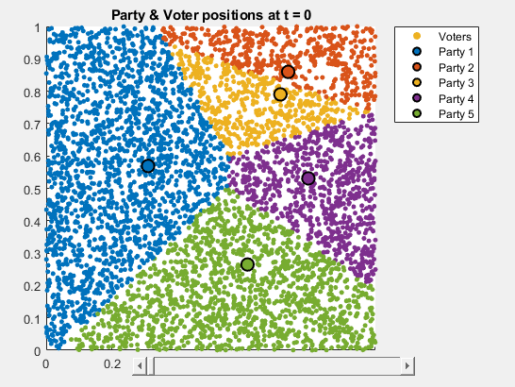
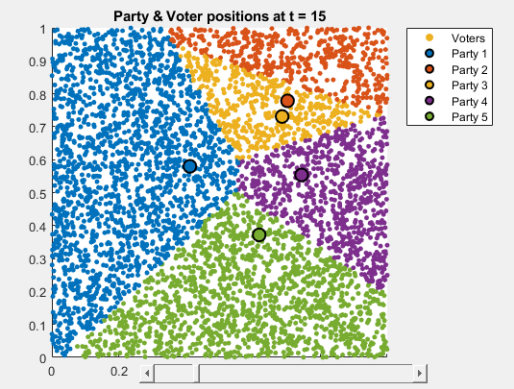


Figure 1: Party Positions Over Time $\eta = 0.1$ $t = 0, 4, 27, 49$.

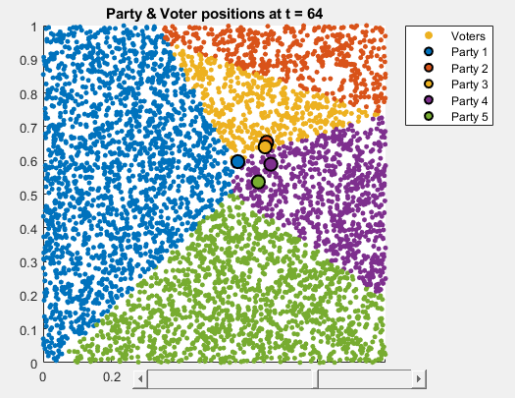
The second parameter, κ, determines how parties update their positions given party competition. This parameter operates as a Peer-Party Influence parameter which moves parties toward the geometric centroid of other parties. The following figures demonstrates this behavior with all other parameters disabled.



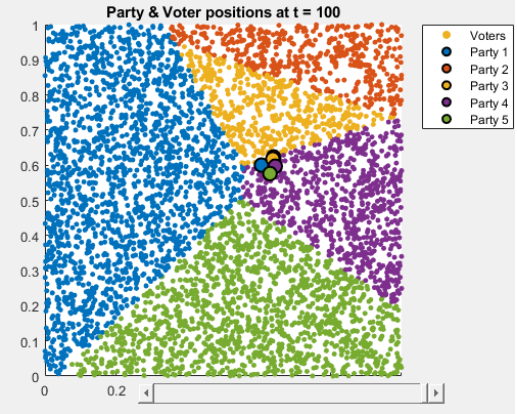
(a) $t = 0$



(b) $t = 15$

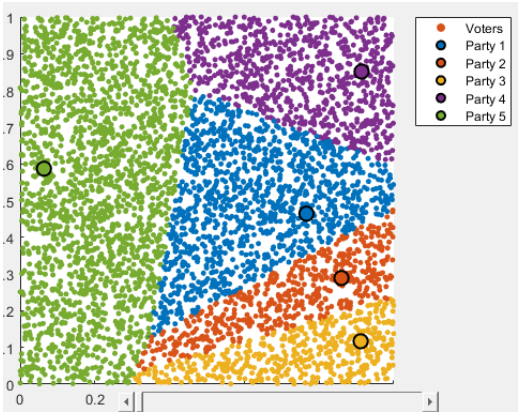


(c) $t = 64$

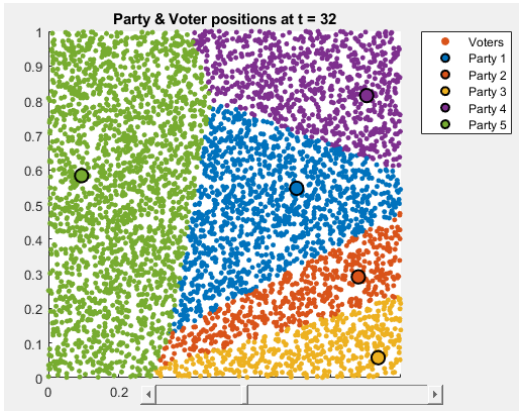


(d) $t = 100$

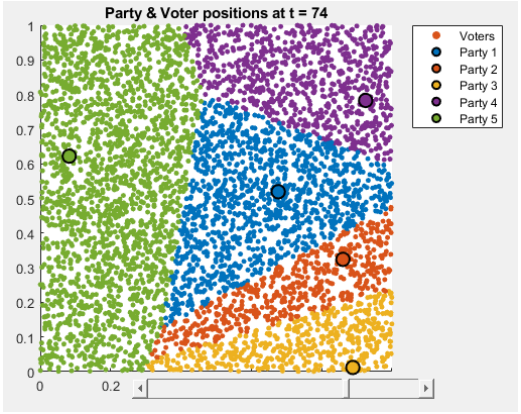
We also added a noise parameter σ. This parameter causes parties to randomly move some amount at each timestep. The following figures demonstrate the behavior of sigma with other parameters disabled.



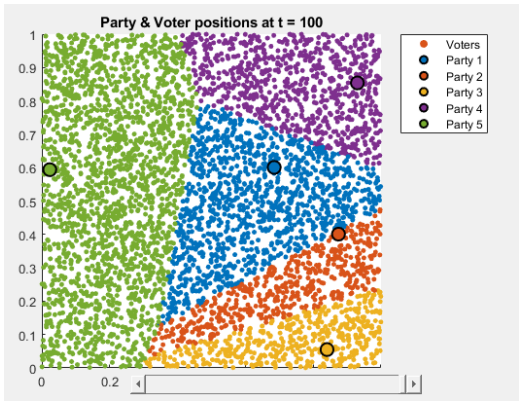
(a) $t = 0$



(b) $t = 32$



(c) $t = 74$



(d) $t = 100$

Finally, we added a repulsion parameter ρ. This parameter weights the repulsion between the two parties which are initially the furthest from each other. This causes the two parties to continue moving away from each other. We believe this behavior makes sense considering that opposing extremist parties may try to distance themselves from each other. The following figures demonstrate the expected behavior for ρ enabled and all other update parameters equal to zero.

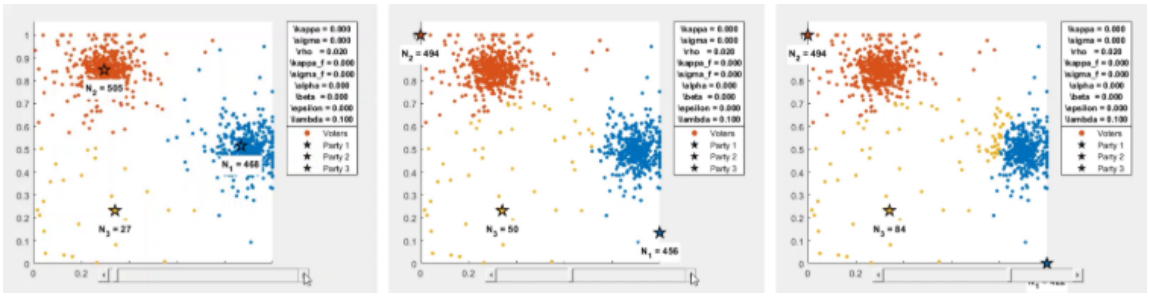
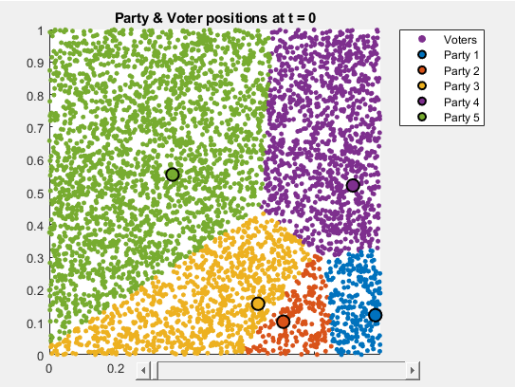


Figure 2: ρ Party Movement Over Time

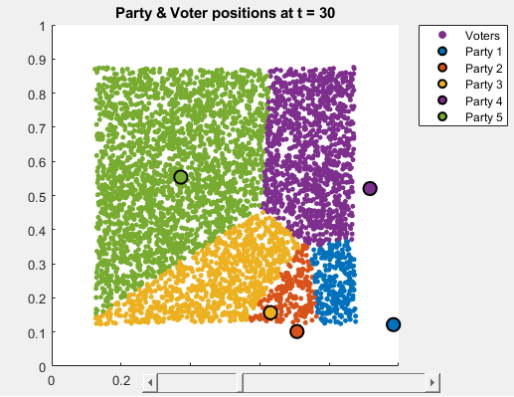
4.2 Individual Update

We have simulated and analyzed the behaviors created by each parameter and mechanism for individuals' position updates. The following sets of figures demonstrate the behavior of each parameter, with other parameters held constant.

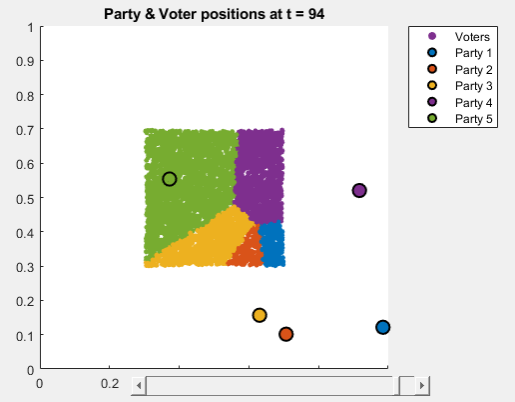
The first parameter for individual update that we created was α. Alpha is a Peer Influence parameter, which is responsible for moving individuals towards the ideological position of other individuals. Alpha is regulated by another parameter ϵ. Epsilon is the active radius for alpha, that is, individuals are attracted alpha amount to individuals within epsilon distance. The following figures demonstrates the behavior of alpha with other parameters disabled.



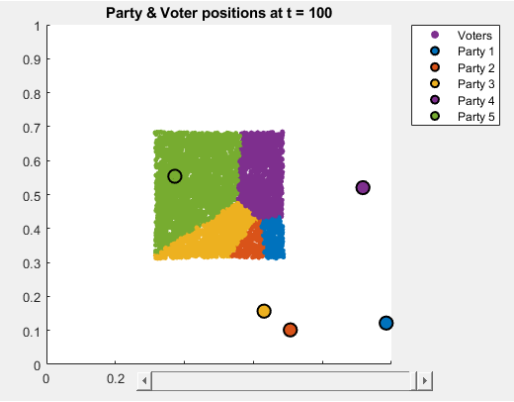
(a) $t = 0$



(b) $t = 30$

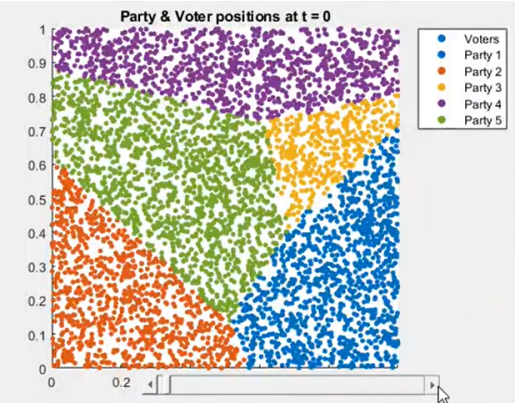


(c) $t = 93$

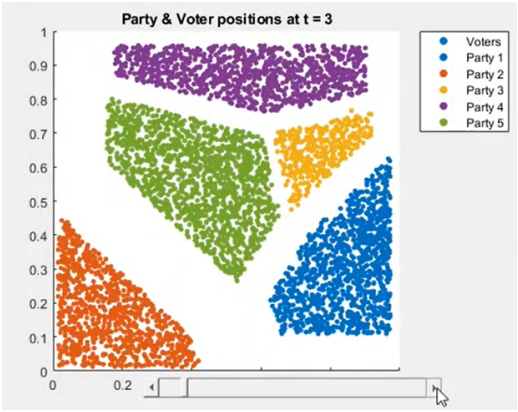


(d) $t = 100$

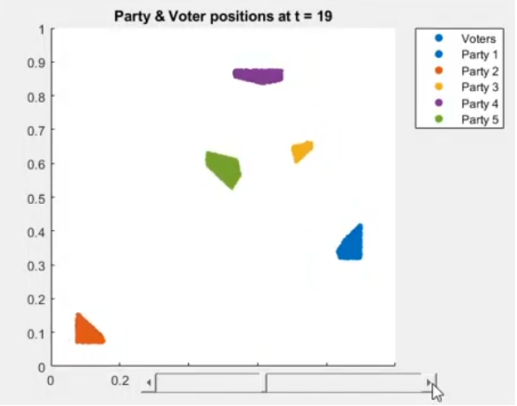
The second parameter for individual update is β. Beta is the Voter's Party Influence parameter and causes voters to move towards the party they voted for. The following figures demonstrates the behavior of beta with other parameters disabled.



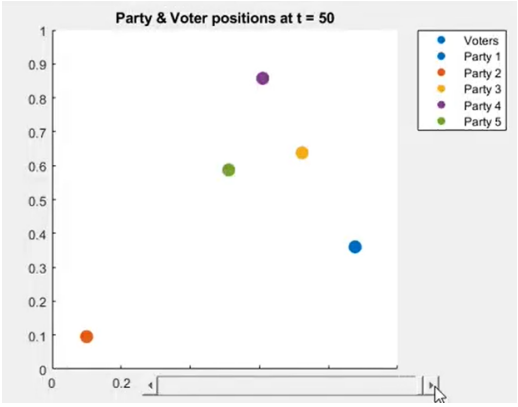
(a) $t = 0$



(b) $t = 3$



(c) $t = 19$



(d) $t = 50$

Figure 3: Party and Individual position updates at different timesteps.

4.3 Voter Satisfaction

We have produced figures for voter satisfaction using the newest voter satisfaction function, which restricts values to within 0 and 10. The following are figures that demonstrate voter satisfaction across time, number of parties, and with different parameters.

In order to understand general effects on voter satisfaction, we examined the independent effects of each parameter on voter satisfaction.

The following figure shows the effects of different alphas on satisfaction over time, with a fixed seed for simulation initialization. We can see that voter satisfaction increases more rapidly with higher values of alpha. Since alpha weights the influence of peers on each other, higher alpha makes individuals move towards each other faster. When epsilon increases, alpha tends to move individuals toward the geometric centroid of all voters.

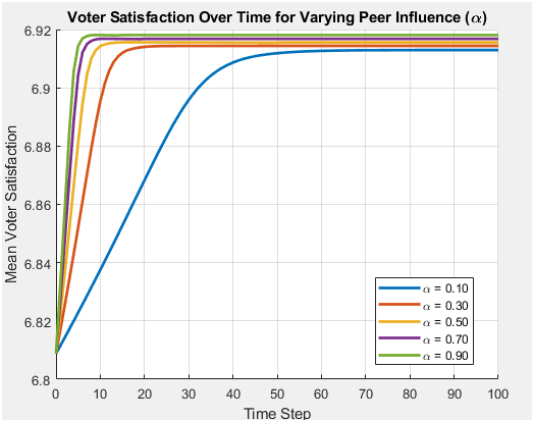


Figure 4: Satisfaction for Different Alpha

The following figure shows the effects of different betas on satisfaction over time, with a fixed seed for simulation initialization. We see that higher beta values cause voter satisfaction to increase more rapidly. Beta causes voters to move toward the ideological position of the party they vote for, therefore it follows that satisfaction should increase since voters become more aligned with their parties.

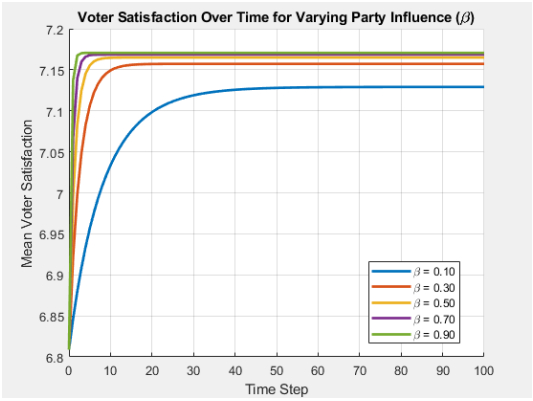


Figure 5: Satisfaction for Different Beta

The following figure shows the effects of different kappas on satisfaction over time, with a fixed seed for simulation initialization. Since beta weights the movement of parties towards each other, typically resulting in movement away from their voter bases, we observe that all values of kappa cause a rapid decrease in voter satisfaction.

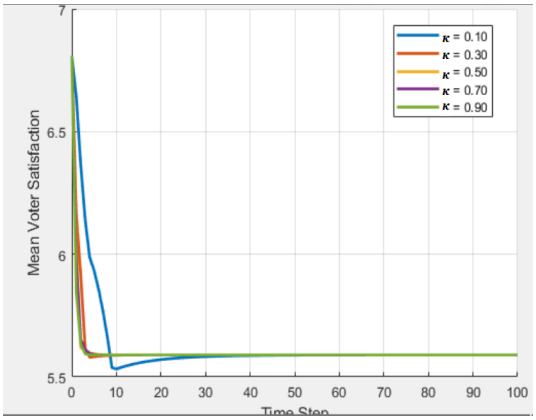


Figure 6: Satisfaction for Different Kappa

The following figure shows the effects of different sigmas on satisfaction over time, with a fixed seed for simulation initialization. Since sigma weights the random movement of parties, we observe that all values of sigma greater than zero cause a random change in satisfaction at each time step.

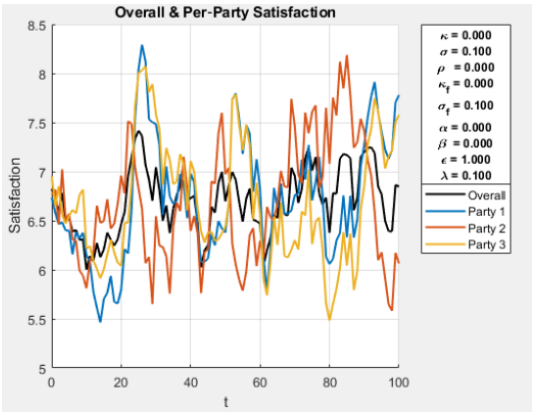


Figure 7: Satisfaction for Different Sigmas

The following figure shows the effects of different rho on satisfaction over time, with a fixed seed for simulation initialization. Since rho weighs the repulsion of the two farthest parties, we should expect it to cause a decrease in satisfaction as the parties should usually move away from their voter bases.

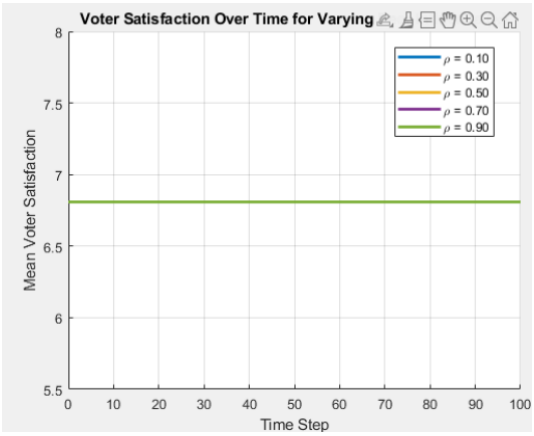


Figure 8: Satisfaction for Different Rhos

While we analyzed most parameters over time, within single simulations with fixed seeds, we analyzed eta slightly different. The following figure shows the final overall satisfaction, averaged across multiple runs, for different number of parties. This also allows us to see the general effect of eta on satisfaction.

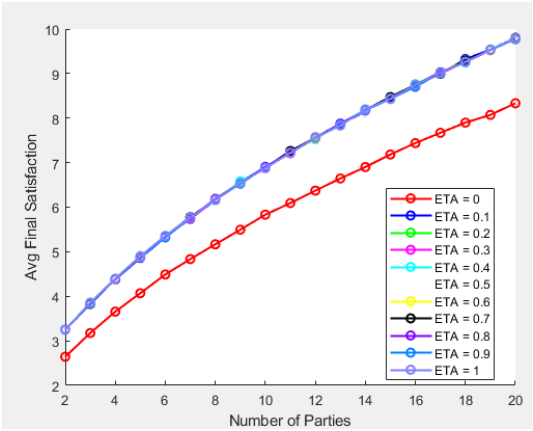


Figure 9: Different ETA Simulation

Our newest measure of voter satisfaction utilizes the parameter lambda in order to regulate which parties contribute to the satisfaction of any given individual. The following figure demonstrates the final overall voter satisfaction over number of parties with different lambda values for each line.

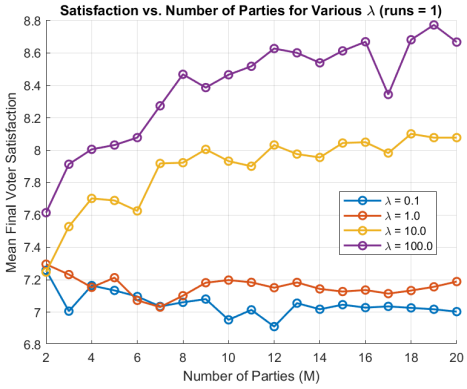


Figure 10: Final Overall Satisfaction x Number of Parties x Lambdas

5 Discussion and Conclusion

We developed our model to analyze how voter and party movement within an ideological space affects voter satisfaction and polarization. By systematically varying initial conditions and model parameters, we uncover clear relationships between those parameters and both satisfaction and polarization outcomes.

Effect of the parameter λ. Voter satisfaction increases as λ grows. In our formulation, higher values of λ place greater weight on the closest parties when computing an individual's satisfaction. In the limit $\lambda \to \infty$, each voter's satisfaction depends entirely on the distance to their nearest party, which maximizes satisfaction because that party is ideologically the most similar.

Impact of the number of parties. When λ is large, satisfaction also rises with the number of competing parties. Intuitively, more parties populate the ideological space more densely, reducing the average distance between any voter and their closest party, and thereby increasing overall satisfaction.

Future extensions. To make the model more realistic, we plan to:

- Introduce additional mechanisms governing how both individuals and parties adjust their positions, including empirically calibrated social and institutional influences.
- Refine the satisfaction function to account for political outcomes—such as which party or coalition governs—and not just ideological proximity.
- Calibrate our update rates $(\eta, \alpha, \beta, \kappa, \sigma)$ using empirical data rather than exploratory tuning.

These enhancements will allow us to simulate and interpret real-world electoral dynamics with greater accuracy.

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